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On functional equations associated with the renormalization of non-commuting circle mappings

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Abstract
We investigate the functional equations associated with critical non-commuting circle homeomorphisms. Using Epstein's Herglotz function methods, we show that there are analytic solutions of a class of functional equations closely related to period-two points of the circle-map renormalization transformations for rotation number with continued fraction $[p, p, \ldots]$, $p \in \mathbb{N}$.

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1. Introduction

Since the 1980s critical circle maps have been extensively studied, not least because they play an important role in the so-called Ruelle-Takens approach to the onset of turbulence. In particular, the appearance of a single critical point in a circle diffeomorphism with irrational rotation number is an idealized model of the break-up of an invariant torus in phase space on which the flow is quasiperiodic. For a review of this theory we refer the reader to [R].

Following the pioneering work of Feigenbaum with regard to period doubling [F1, F2], universal behaviour for critical circle maps [FKS] was explained by a renormalization analysis [ORSS, FKS], which was based on the existence and hyperbolicity of fixed points of a renormalization transformation. The universal scaling behaviour is governed by a so-called trivial fixed point in the case of circle diffeomorphisms and by a non-trivial critical fixed point for critical circle maps. For the case of golden-mean rotation number, the existence of the non-trivial fixed point was established first analytically for degree close to 1 [JR], second for cubic maps by computer-assisted means [M] and finally analytically for all degrees [EE]. Complex analytic methods have also been used by Yampolsky to establish strong convergence results

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for odd integer degree [Y1, Y2]. Recently, Yampolsky has constructed a renormalization horseshoe for circle maps [Y3].

In [MO1], the theory for golden-mean circle diffeomorphisms was extended to the case of 'almost-C^1 maps'. The scaling for such maps was found to be governed by period-two points of the golden-mean circle renormalization transformation and to depend on a 'modulus' which determines the universality class of the circle map. Further, more general results in this area have been obtained by Khanin and coworkers [KV, KK].

In this paper we begin the extension of the results in [MO1] to the case of critical circle maps, and for the case of rotation number ρ with continued fraction expansion $[p, p, p, \ldots]$ for $p \geqslant 1$ an integer.

Recall that the dynamics of a circle homeomorphism $h : \mathbb{T}^1 \to \mathbb{T}^1$ are determined by the arithmetic properties of the rotation number $\rho = \rho(h) = \lim_{n\to\infty}(f^n(x) - x)/n \bmod 1$, where $f : \mathbb{R} \to \mathbb{R}$ is a lift of h to $\mathbb{R}$. In particular ρ is rational iff h has a periodic orbit and (for h sufficiently smooth) ρ is irrational iff h has a dense orbit. The scaling properties of the regions in function space (the Arnol'd tongues) are governed by the renormalization transformation acting on pairs of maps (ξ, η) that glue together to make a circle homeomorphism. Specifically, let $1/(p + 1) < \rho(h) < 1/p$ for $p \geqslant 1$ a positive integer, and let ξ and η be increasing maps defined on intervals containing $[\eta(0), 0]$ and $[0, \xi(0)]$ respectively, and such that

(1) $\eta(\xi(0)) = \xi(\eta(0))$,
(2) $0 < \xi(0) < 1$,
(3) $\xi^p(\eta(0)) > 0$,
(4) $\xi^{p-1}(\eta(0)) < 0$.

Then defining h by

$$h(x) = \begin{cases} \xi(x), & \eta(0) \leqslant x \leqslant 0 \\ \eta(x), & 0 \leqslant x \leqslant \xi(0) \end{cases}$$

gives a homeomorphism h on the circle obtained by identifying the points $\xi(0)$ and $\eta(0)$. The rotation number $\rho(\xi, \eta)$ of the pair (ξ, η) is then $\rho(h)$.

Using juxtaposition to denote function composition and scaling, the renormalization transformation T_p on function pairs (ξ, η) is defined by

$$T_p(\xi, \eta) = (\beta^{-1}\xi^{p-1}\eta\beta, \beta^{-1}\xi^{p-1}\eta\xi\beta), \qquad \beta < 0$$

and satisfies

$$\rho(T_p(\xi, \eta)) = \frac{1}{\rho(\xi, \eta)} - p.$$

(See [ORSS] for further details.)

Let $r \geqslant 1$ be an odd integer. (In our subsequent work we shall take $r \geqslant 1$ real, but for simplicity we consider only odd integers here.) Our work in this paper concerns the question of the existence of a family of period-two points of the transformation T_p.

The origin of the family may be readily understood in terms of the renormalization theory of commuting circle-map pairs, i.e, pairs satisfying $\xi\eta = \eta\xi$ (at least on a neighbourhood of 0). Restricted to the space of commuting pairs (ξ, η), the transformation T is hyperbolic at each fixed point of degree $r \geqslant 1$, each with a single essential eigenvalue $\delta > 1$. However, once commutativity is no longer imposed, hyperbolicity is lost and, as observed in [ORSS], an eigenvalue -1 appears. This eigenvalue indicates the existence of a line of period-two points of T, corresponding to non-commuting circle-map pairs.

Indeed, a heuristic analysis of the spectrum of the derivative of the renormalization operator T_p at the degree-r fixed point (the existence of which was proved by Eckmann and

Epstein [EE]) shows that there is an eigenvalue -1 corresponding to infinitesimal perturbations of the fixed point that satisfy $(\xi\eta-\eta\xi)(0)=0$, $(\xi\eta-\eta\xi)'(0)=0,\ldots,(\xi\eta-\eta\xi)^{(r-1)}(0)=0$, but $(\xi\eta-\eta\xi)^{(r)}(0)\neq 0$, so that the commuting condition is satisfied up to $(r-1)$ th order but not up to rth order. (We refer the reader to [ORSS] for details of these calculations.) This result suggests strongly the existence of a one-parameter family of period-two points through the degree-r fixed point corresponding to breaking the commuting symmetry $\xi\eta=\eta\xi$ on a neighbourhood of 0, so that, for all $p\geqslant 1$, and for all $r>1$, there exist pairs (ξ,η), $(\tilde{\xi},\tilde{\eta})$ such that $T_p(\xi,\eta)=(\tilde{\xi},\tilde{\eta})$ and $T_p(\tilde{\xi},\tilde{\eta})=(\xi,\eta)$ with critical points of degree r at 0.

We now define the modulus $s=s(\xi,\eta)$ by

$$s=\frac{(\eta\xi)^{(r)}(0)}{(\xi\eta)^{(r)}(0)}. \tag{1.1}$$

A straightforward calculation shows that T_p inverts the modulus s, i.e., $s(T_p(\xi,\eta))=s(\xi,\eta)^{-1}$. This means that a fixed-point pair of T_p necessarily has $s(\xi,\eta)=1$. For $s(\xi,\eta)\neq 1$, we will have $s(T_p^2(\xi,\eta))=s(\xi,\eta)$, which is certainly consistent with a period-two point of T_p.

In [MO1], Mestel and Osbaldestin studied non-commuting circle-map pairs, in the context of understanding the scaling behaviour of implicit complex maps on the boundary of a golden-mean Siegel disc. It was observed that a line of period-two points did indeed exist through the trivial fixed point (ξ_L,η_L) (the case $r=1$ here). The period-two points were given by fractional-linear maps and were parametrized by an invariant 'modulus' μ given for a pair (ξ,η) by

$$\mu=\frac{(\eta\xi)'(0)}{(\xi\eta)'(0)}. \tag{1.2}$$

Similar results have been obtained by Khanin and Vul [KV]. Formula (1.2) has a natural generalization to degree-r maps given by (1.1) above.

Thus, our aim is working towards the proof of the following conjecture, which follows on from previous work of Khanin and Vul [KV] and of Mestel and Osbaldestin [MO1] on non-commuting almost-C^1 maps, and of the numerical observations in [Z].

Let us now extend $r>1$ to real values as indicated above. We consider pairs (ξ,η) satisfying (1)–(4) above and which may be written as functions of $x^{(r)}=x|x|^{r-1}$. The renormalization transformation T_p given above preserves the space of pairs of this form.

Writing $\xi(x)=E(x^{(r)})$, $\eta(x)=F(x^{(r)})$ where E and F are C^1 at 0, the modulus s may be written in terms of E and F as follows:

$$s(E,F)=\frac{F'(E(0)^{(r)})|E(0)|^{r-1}E'(0)}{E'(F(0)^{(r)})|F(0)|^{r-1}F'(0)}. \tag{1.3}$$

This formula extends to all real values of $r>1$.

Conjecture

(1) (Existence) For all $r>1$ and all $\mu>0$ there exists a solution pair (ξ,η) to the equation $T_p^2(\xi,\eta)=(\xi,\eta)$, with $\xi(x)=E(x^{(r)})$, $\eta(x)=F(x^{(r)})$, with $E'(0),F'(0)\neq 0$ and with $s(\xi,\eta)=s(E,F)=\mu$. Furthermore E and F are analytic on a neighbourhood of 0.

(2) (Hyperbolicity) Restricted to the space of pairs (ξ,η) with r fixed and $s(\xi,\eta)=\mu$, the period-two orbit is hyperbolic with a single unstable direction, i.e., the spectrum of $dT_p^2(\xi,\eta)$ consists of a single eigenvalue Δ, with $|\Delta|>1$, and all other eigenvalues lie strictly within the unit circle.

The conjecture is in line with the results for commuting circle maps [EE, L], and suggests that the universal scaling of non-commuting circle maps is governed by a non-trivial period-two point of the relevant renormalization transformation and that the universality class of critical circle maps is dependent on two parameters, namely, the degree $r > 1$ of the critical point and, a second, asymmetry parameter, or 'modulus' given for r an odd integer by equation (1.1).

In this paper we make considerable progress in the first conjecture. A period-two point (ξ, η) with image $(\tilde{\xi}, \tilde{\eta})$ under T_p satisfies the equations

$$\tilde{\xi} = \beta^{-1}\xi^{p-1}\eta\beta, \qquad \tilde{\eta} = \beta^{-1}\xi^{p-1}\eta\xi\beta \tag{1.4}$$

$$\xi = \tilde{\beta}^{-1}\tilde{\xi}^{p-1}\tilde{\eta}\tilde{\beta}, \qquad \eta = \tilde{\beta}^{-1}\tilde{\xi}^{p-1}\tilde{\eta}\tilde{\xi}\tilde{\beta} \tag{1.5}$$

which, on elimination of η, and $\tilde{\eta}$ gives

$$\xi = \tilde{\beta}^{-1}\tilde{\xi}^{p}\beta^{-1}\xi\beta\tilde{\beta}, \qquad \tilde{\xi} = \beta^{-1}\xi^{p}\tilde{\beta}^{-1}\tilde{\xi}\beta\tilde{\beta}. \tag{1.6}$$

To prove the first conjecture it would be sufficient to solve (1.6), since a solution of (1.6) readily provides a solution of (1.4). Instead we obtain a solution of the following equations:

$$\xi = \tilde{\beta}^{-1}\tilde{\xi}^{p}\beta^{-1}\xi(\beta\tilde{\beta})^{\rho/r}, \qquad \xi = \beta^{-1}\xi^{p}\tilde{\beta}^{-1}\tilde{\xi}(\beta\tilde{\beta})^{\tilde{\rho}/r}, \tag{1.7}$$

where $\rho, \tilde{\rho} > 0$ satisfy $\max(\rho, \tilde{\rho}) = r$, but are otherwise undetermined. Of course, this includes (1.6) as a special case. Despite the sophistication of the mathematical techniques involved, further work is needed to establish the precise equations (1.6) and parametrization required for the circle-map renormalization. Nevertheless we believe that we have made a significant step forward in this theory.

In the spirit of the work of Mestel and Osbaldestin on asymmetric period-doubling renormalization [MO2], we use Herglotz function methods. We recast the renormalization equations in terms of Herglotz functions and, using techniques of Epstein [E2], we show that there is a one-parameter family of solutions to a class of associated functional equations.

The remainder of this paper is organized as follows. In section 2, we reformulate the circle-map renormalization fixed-point problem so that Herglotz-function methods may be utilized. A statement of our results is given in section 3. Following preliminary material on Herglotz functions in section 4, we prove our results in section 5. Finally, in section 6, we make some concluding remarks.

2. Reformulation of a circle-map problem

We now reformulate briefly the functional equations for period-two points of the operator T_p so that the Herglotz-function method may be employed.

A period-two point of the renormalization transformation T_p, for integer $p \geqslant 1$ satisfies the following equations:

$$(\tilde{\xi}, \tilde{\eta}) = T_p(\xi, \eta) = (\beta^{-1}\xi^{p-1}\eta\beta, \beta^{-1}\xi^{p-1}\eta\xi\beta), \qquad \beta < 0 \tag{2.1}$$

$$(\xi, \eta) = T_p(\tilde{\xi}, \tilde{\eta}) = (\tilde{\beta}^{-1}\tilde{\xi}^{p-1}\tilde{\eta}\tilde{\beta}, \tilde{\beta}^{-1}\tilde{\xi}^{p-1}\tilde{\eta}\tilde{\xi}\tilde{\beta}), \qquad \tilde{\beta} < 0 \tag{2.2}$$

with $0 < \beta\tilde{\beta} < 1$ and with normalizations $\xi(0) = \tilde{\xi}(0) = 1$. These equations may be readily reformulated so that the Herglotz-function method may be used. Indeed, we may first eliminate the functions $\eta, \tilde{\eta}$ to obtain

$$\xi = \tilde{\beta}^{-1}\tilde{\xi}^{p}\beta^{-1}\xi\beta\tilde{\beta}, \qquad \tilde{\xi} = \beta^{-1}\xi^{p}\tilde{\beta}^{-1}\tilde{\xi}\beta\tilde{\beta}, \tag{2.3}$$

and then, writing $\xi(x) = E(x|x|^{r-1}), \tilde{\xi}(x) = \tilde{E}(x|x|^{r-1}), U(x) = -E^{-1}(x), \tilde{U}(x) = -\tilde{E}^{-1}(x), \lambda = -\beta > 0, \tilde{\lambda} = -\tilde{\beta} > 0$, we obtain

$$U(x) = (\lambda\tilde{\lambda})^{-r} U(\tilde{\phi}(x)), \qquad \tilde{U}(x) = (\lambda\tilde{\lambda})^{-r}\tilde{U}(\phi(x)), \tag{2.4}$$

where

$$\phi(x) = \tilde{\lambda}\hat{V}^p(\lambda x), \qquad \tilde{\phi}(x) = \lambda\tilde{\hat{V}}^p(\tilde{\lambda}x), \tag{2.5}$$

and $\hat{V}(x) = U(-x)^{1/r}, \tilde{\hat{V}}(x) = \tilde{U}(-x)^{1/r}$. We note that we have the normalizations $U(1) = \tilde{U}(1) = 0$. Further normalizing by writing $\psi(x) = U(x)/U(0), \tilde{\psi}(x) = \tilde{U}(x)/\tilde{U}(0)$, we obtain the functional equations

$$\psi(x) = (\lambda\tilde{\lambda})^{-r}\psi(\tilde{\phi}(x)), \qquad \tilde{\psi}(x) = (\lambda\tilde{\lambda})^{-r}\tilde{\psi}(\phi(x)), \tag{2.6}$$

where $\psi(0) = \tilde{\psi}(0) = 1$ and $\psi(1) = \tilde{\psi}(1) = 0$ and where

$$\phi(x) = \tilde{\lambda}(z_1\tilde{\lambda}^{-1}v)^p(\lambda x), \qquad \tilde{\phi}(x) = \lambda(\tilde{z}_1\lambda^{-1}\tilde{v})^p(\tilde{\lambda}x), \tag{2.7}$$

and $v(x) = \psi(-x)^{1/r}, \tilde{v}(x) = \tilde{\psi}(-x)^{1/r}$. These are equations of the general form (2.6), (2.7) that we are concerned with in this paper.

3. Statement of the main results

We now give the main result that we prove in our paper.

Theorem 1. *Let $r > 1, \gamma > 0$ be fixed real numbers, and p a positive integer. Then, there exist real numbers $\lambda > 0, \tilde{\lambda} > 0, \rho > 0, \tilde{\rho} > 0, a \in (0,1), \tilde{a} \in (0,1)$, with $\lambda\tilde{\lambda} \in (0,1), \tilde{\lambda}/\lambda = \gamma, \max(\rho, \tilde{\rho}) = r$, and anti-Herglotz functions $\psi, \tilde{\psi}$, analytic on $(-\tilde{\lambda}^{-1}, \tilde{a}^{-1}), (-\lambda^{-1}, a^{-1})$ respectively, such that*

$$\psi(z) = \frac{1}{(\lambda\tilde{\lambda})^{\tilde{\rho}}}\psi(\tilde{\phi}(z)), \qquad \psi(0) = 1, \quad \psi(1) = 0, \tag{3.1}$$

$$\tilde{\psi}(z) = \frac{1}{(\lambda\tilde{\lambda})^{\rho}}\tilde{\psi}(\phi(z)), \qquad \tilde{\psi}(0) = 1, \quad \tilde{\psi}(1) = 0, \tag{3.2}$$

where

$$\phi(z) = \tilde{\lambda}(z_1\tilde{\lambda}^{-1}v)^p(\lambda z), \tag{3.3}$$

$$\tilde{\phi}(z) = \lambda(\tilde{z}_1\lambda^{-1}\tilde{v})^p(\tilde{\lambda}z), \tag{3.4}$$

where

$$v(z) = \psi(-z)^{\frac{1}{r}}, \tag{3.5}$$

$$\tilde{v}(z) = \tilde{\psi}(-z)^{\frac{1}{r}}, \tag{3.6}$$

and the real numbers $z_1 \in (0,1), \tilde{z}_1 \in (0,1)$ are chosen so that $\phi(1) = 1, \tilde{\phi}(1) = 1$.

Comparing equations (3.1), (3.2) with (2.6), we see that they correspond precisely when $\rho = \tilde{\rho} = r$. This is equivalent to the condition that $\phi'(1) = \tilde{\phi}'(1)$. Furthermore, the closer is the agreement between $\phi'(1)$ and $\tilde{\phi}'(1)$, the closer is the agreement between ρ and $\tilde{\rho}$. Thus equation (2.6) involves a further symmetry that it is not straightforward to impose on the solution. This problem appears to be technical, rather than fundamental. A further difference is that (again for technical reasons) we have chosen to use the ratio $\gamma = \tilde{\lambda}/\lambda$ to parametrize the family of solutions, rather than μ. Nevertheless, in view of the difficulty of the nonlinear functional equations, our result constitutes a significant progress towards establishing the existence of solutions of (2.6).

In the remainder of this paper, we give the proof of the results outlined above. The methods follow those of Epstein in [E2], although there are many differences in our approach. It turns out that the case $p = 1$ is simpler and serves as an introduction to the more complicated case $p \geqslant 2$, which requires a more careful analysis.

We now present some important background results on Herglotz functions which we shall use extensively in the subsequent sections.

4. Herglotz function preliminaries and notation

We denote $\mathbb{C}_+$ the upper-half complex plane $\{z : \operatorname{Im} z > 0\}$, $\mathbb{C}_- = -\mathbb{C}_+$, and $\overline{\mathbb{C}}_+, \overline{\mathbb{C}}_-$ the closures of $\mathbb{C}_+, \mathbb{C}_-$, respectively. If $I \subseteq \mathbb{R}$ is an open interval, we denote $\Omega(I)$ the domain $\Omega(I) = \mathbb{C}_+ \cup \mathbb{C}_- \cup I$.

A function F is a Herglotz function if it is analytic in $\mathbb{C}_+ \cup \mathbb{C}_-$, $F(\mathbb{C}_+) \subset \overline{\mathbb{C}}_+$, and $F(\mathbb{C}_-) \subset \overline{\mathbb{C}}_-$. Similarly, a function ψ is an anti-Herglotz function if it is analytic in $\mathbb{C}_+ \cup \mathbb{C}_-$, $\psi(\mathbb{C}_+) \subset \overline{\mathbb{C}}_-$, and $\psi(\mathbb{C}_-) \subset \overline{\mathbb{C}}_+$.

If a non-constant Herglotz function F is also analytic on an interval $I \subseteq \mathbb{R}$, then F is strictly increasing and has a positive Schwarzian derivative $S(F) = (F''/F')' - (F''/F')^2/2$ on I [D]. One consequence of this is that F cannot have a local maximum in its first derivative. Similarly, if a non-constant anti-Herglotz function ψ is also analytic on an interval $I \subseteq \mathbb{R}$, then ψ is strictly decreasing and has a positive Schwarzian derivative $S(\psi)$ on I. On any interval I on which a Herglotz or anti-Herglotz function is analytic, the right- and left-hand limits exist respectively at the left- and right-hand endpoints of I, although they may be $\pm\infty$, and therefore the function can be continuously extended to $\bar{I}$.

For any two real numbers A, B, with $A < 0 < 1 < B$, we denote $H(A, B)$, $AH(A, B)$ the spaces of Herglotz and anti-Herglotz functions respectively which are also analytic on the interval (A, B) on the real axis. We equip $H(A, B)$ and $AH(A, B)$ with the topology of uniform convergence on compact subsets of $\Omega(A, B)$.

Any Herglotz function F admits the integral representation [H, AG, D]

$$F(z) = c_1 + c_2 z + \int_{\mathbb{R}} \left\{ \frac{1}{t - z} - \frac{t}{t^2 + 1} \right\} \mathrm{d}\rho(t), \tag{4.1}$$

where c_1, c_2 are real constants ($c_2 \geqslant 0$), and the function $\rho(t)$ is non-decreasing, right continuous, defined up to an additive constant, and satisfies the convergence condition

$$\int_{\mathbb{R}} \frac{1}{1 + t^2} \mathrm{d}\rho(t) < +\infty. \tag{4.2}$$

For given F, the constants c_1, c_2 are determined by $c_1 = \operatorname{Re} F(i)$, $c_2 = \lim_{s \to +\infty} \frac{1}{s} \operatorname{Im} F(\mathrm{i}s)$, $(s \in \mathbb{R})$, and ρ gives rise to a measure μ through the relation $\rho(b) - \rho(a) = \mu((a, b])$ for finite intervals $(a, b]$. We refer to μ as the Herglotz measure associated with F.

The boundary value $F_+(x)$ of F at a point x on the real axis is defined by $F_+(x) = \lim_{\varepsilon \to 0^+} F(x + \mathrm{i}\varepsilon) (x \in \mathbb{R}, \varepsilon \in \mathbb{R})$. $F_+(x)$ exists Lebesgue-almost everywhere either as a real number or as a complex number with a strictly positive imaginary part. The absolutely continuous part $\mu_{\text{a.c.}}$ of μ is concentrated on the set of points $\{x \in \mathbb{R} : \operatorname{Im} F_+(x) > 0\}$, and the density function of $\mu_{\text{a.c.}}$ is given by $\frac{1}{\pi} \operatorname{Im} F_+(x)$. See [P] for an analysis of the support of the absolutely continuous and singular parts of μ. Hence the measure associated with the Herglotz function $\log F(z)$ is purely absolutely continuous, and can be written as $\sigma(t)\,\mathrm{d}t$, where the density function $\sigma(t)$ is defined almost everywhere and satisfies $0 \leqslant \sigma(t) \leqslant 1$.

For any two real numbers $m_1 > 0, m_2 > 0$ we denote $E(m_1, m_2)$ the space of anti-Herglotz functions ψ analytic in the domain $\Omega(-1/m_1, 1/m_2)$, and such that $\psi(0) = 1, \psi(1) = 0$. Then, such a function has the representation

$$\log \psi(z) = \int_{\mathbb{R}\setminus(-1/m_1,1)} \left\{\frac{1}{t} - \frac{1}{t-z}\right\}\sigma(t)\,\mathrm{d}t, \qquad \forall z \in \Omega(-1/m_1, 1), \tag{4.3}$$

where $0 \leqslant \sigma(t) \leqslant 1, \sigma(t) = 0$ for all $t \in [-1/m_1, 1]$, and $\sigma(t) = 1$ for all $t \in [1, 1/m_2]$. It follows from the above representation that ψ satisfies the following inequalities [E1, E2]:

$$\frac{\psi(z)(1-m_2 z)}{1-z} \leqslant 1 \leqslant \frac{\psi(z)(1+m_1 z)}{1-z}, \qquad \forall z \in (0, 1/m_2)\setminus\{1\}, \tag{4.4}$$

reversed for $z \in (-1/m_1, 0)$,

$$\frac{1-m_2}{(1-z)(1-m_2 z)} \leqslant -\frac{\psi'(z)}{\psi(z)} \leqslant \frac{1+m_1}{(1-z)(1+m_1 z)}, \qquad \forall z \in (-1/m_1, 1/m_2)\setminus\{1\}. \tag{4.5}$$

Moreover, if $\psi(-1/m_1) \leqslant M$, for some (positive) constant M, then

$$-\frac{\psi'(z)}{\psi(z)} \leqslant \frac{\log M}{(-4z)(1+m_1 z)}. \tag{4.6}$$

We shall also need the following lemma which is proved in [E2].

Lemma 1

(i) Let $A < a < b < B$ be real numbers, and $F \in H(A, B)$. Then, for each $z \in (a, b)$,

$$F(z) \geqslant \frac{(B-b)(z-a)F(b) + (B-a)(b-z)F(a)}{(b-a)(B-z)}. \tag{4.7}$$

(ii) Let A, B, A', B' be strictly positive real numbers. If F is an analytic map of $\Omega(-A, B)$ into $\Omega(-A', B')$ with $F(0) = 0$, then

$$|F'(0)| \leqslant \frac{A'B'(A+B)}{AB(A'+B')}. \tag{4.8}$$

5. Proof of the principal theorem

5.1. The case $p = 1$

In this section $p = 1$, and $r > 1, \gamma > 0$ are fixed real numbers. The real numbers Λ_-, Λ_+, with $0 < \Lambda_- < \Lambda_+ < 1$, are also fixed, but will be chosen later. Let $b = \sqrt{\frac{\Lambda_+}{\gamma}}, \tilde{b} = \sqrt{\gamma\Lambda_+}$. We note that $b\tilde{b} = \Lambda_+$. For any two strictly positive real numbers s, t, let a function h be defined by

$$h_{s,t}(z) = \frac{z(s+1)}{z(s-t)+1+t}, \tag{5.1}$$

and note that $h_{s,t}(0) = 0, h_{s,t}(1) = 1$, and $h_{s,t}(-1/s) = -1/t$.

We denote $\mathcal{Q}_1(r, \gamma, \Lambda_+)$ the space of all pairs of functions $(\Phi, \tilde{\Phi})$ with the following properties:

(q1) $\Phi \in H(-b^{-1}, \Lambda_+^{-1}), \tilde{\Phi} \in H(-\tilde{b}^{-1}, \Lambda_+^{-1})$,
(q2) $\Phi(-b^{-1}, \Lambda_+^{-1}) \subseteq (0, \Lambda_+^{-1}), \tilde{\Phi}(-\tilde{b}^{-1}, \Lambda_+^{-1}) \subseteq (0, \Lambda_+^{-1})$,

(q3) $\Phi(1) = 1, \tilde{\Phi}(1) = 1, \Phi'(1) \leqslant \Lambda_+^r, \tilde{\Phi}'(1) \leqslant \Lambda_+^r, \Phi'(1)\tilde{\Phi}'(1) \geqslant \Lambda_-^{2r}$.

We shall define a continuous operator $B(r, \gamma, \Lambda_+)$ by describing its action on an arbitrary $(\Phi_0, \tilde{\Phi}_0) \in \mathcal{Q}_1(r, \gamma, \Lambda_+)$. We first define functions $\phi_0, \tilde{\phi}_0$ and their linearizers $\psi, \tilde{\psi}$, respectively. Then, we proceed to define the associated functions $\phi, \tilde{\phi}$, and obtain bounds for $\phi'(1), \tilde{\phi}'(1)$.

5.1.1. The functions $\phi, \tilde{\phi}$. Given $(\Phi_0, \tilde{\Phi}_0) \in \mathcal{Q}_1(r, \gamma, \Lambda_+)$, we denote $\Lambda_0 = \Phi_0'(1)^{\frac{1}{r}}, \tilde{\Lambda}_0 = \tilde{\Phi}_0'(1)^{\frac{1}{r}}$. By (q3), $\Lambda_0 \leqslant \Lambda_+, \tilde{\Lambda}_0 \leqslant \Lambda_+$ and $\Lambda_0\tilde{\Lambda}_0 \geqslant \Lambda_-^2$. We set $\Lambda = \min\{\Lambda_0, \tilde{\Lambda}_0\}, \lambda = \sqrt{\frac{\Lambda}{\gamma}}, \tilde{\lambda} = \sqrt{\gamma\Lambda}, \rho = \frac{r \log \Lambda_0}{\log \Lambda} \leqslant r$ and $\tilde{\rho} = \frac{r \log \tilde{\Lambda}_0}{\log \Lambda} \leqslant r$. Note that $\Lambda = \lambda\tilde{\lambda}, 0 < \lambda \leqslant b, 0 < \tilde{\lambda} \leqslant \tilde{b}$, hence $\Lambda \leqslant \Lambda_+$. We define functions $\phi_0, \tilde{\phi}_0$ by

$$\phi_0 = h_{b,\lambda} \circ \Phi_0 \circ h_{b,\lambda}^{-1}, \qquad \tilde{\phi}_0 = h_{\tilde{b},\tilde{\lambda}} \circ \tilde{\Phi}_0 \circ h_{\tilde{b},\tilde{\lambda}}^{-1}. \tag{5.2}$$

If $\lambda = b$, $h_{b,\lambda}$ is the identity. Otherwise, since $\lambda < b$, its pole is below $-b^{-1}$ and $h_{b,\lambda}$ maps $\Omega(-b^{-1}, \Lambda_+^{-1})$ onto $\Omega(-\lambda^{-1}, a_1(\lambda)^{-1})$ where

$$\begin{aligned} &a_1(\lambda)^{-1} = h_{b,\lambda}(\Lambda_+^{-1}),\\ &\lambda\tilde{\lambda} \leqslant \Lambda_+ \leqslant a_1(\lambda) = \frac{b + \Lambda_+ - \lambda(1 - \Lambda_+)}{1 + b} < a_1(0) = \frac{b + \Lambda_+}{1 + b}. \end{aligned} \tag{5.3}$$

Similarly, if $\tilde{\lambda} = \tilde{b}$, $h_{\tilde{b},\tilde{\lambda}}$ is the identity. Otherwise, its pole is below $-\tilde{b}^{-1}$ and $h_{\tilde{b},\tilde{\lambda}}$ maps $\Omega\left(-\tilde{b}^{-1}, \Lambda_+^{-1}\right)$ onto $\Omega(-\tilde{\lambda}^{-1}, \tilde{a}_1(\tilde{\lambda})^{-1})$ where

$$\begin{aligned} &\tilde{a}_1(\tilde{\lambda})^{-1} = h_{\tilde{b},\tilde{\lambda}}\left(\Lambda_+^{-1}\right),\\ &\lambda\tilde{\lambda} \leqslant \Lambda_+ \leqslant \tilde{a}_1(\tilde{\lambda}) = \frac{\tilde{b} + \Lambda_+ - \tilde{\lambda}(1 - \Lambda_+)}{1 + \tilde{b}} < \tilde{a}_1(0) = \frac{\tilde{b} + \Lambda_+}{1 + \tilde{b}}. \end{aligned} \tag{5.4}$$

The functions $\phi_0, \tilde{\phi}_0$ possess the following properties:

(q′1) $\phi_0 \in H(-\lambda^{-1}, a_1(\lambda)^{-1}), \tilde{\phi}_0 \in H(-\tilde{\lambda}^{-1}, \tilde{a}_1(\tilde{\lambda})^{-1})$,
(q′2) $\phi_0(-\lambda^{-1}, a_1(\lambda)^{-1}) \subseteq (0, a_1(\lambda)^{-1}), \tilde{\phi}_0(-\tilde{\lambda}^{-1}, \tilde{a}_1(\tilde{\lambda})^{-1}) \subseteq (0, \tilde{a}_1(\tilde{\lambda})^{-1})$,
(q′3) $\phi_0(1) = 1, \tilde{\phi}_0(1) = 1, \phi_0'(1) = (\lambda\tilde{\lambda})^{\rho}, \tilde{\phi}_0'(1) = (\lambda\tilde{\lambda})^{\tilde{\rho}}$.

We denote $\psi, \tilde{\psi}$ the linearizers of $\phi_0, \tilde{\phi}_0$ respectively, normalized by the condition $\psi(0) = \tilde{\psi}(0) = 1$. Thus, $\psi \in AH(-\tilde{\lambda}^{-1}, \tilde{a}_1(\tilde{\lambda})^{-1}), \tilde{\psi} \in AH(-\lambda^{-1}, a_1(\lambda)^{-1})$, and they satisfy the following equations:

$$\psi(z) = \frac{1}{(\lambda\tilde{\lambda})^{\tilde{\rho}}}\psi(\tilde{\phi}_0(z)), \qquad z \in \Omega(-\tilde{\lambda}^{-1}, \tilde{a}_1(\tilde{\lambda})^{-1}), \quad \psi(0) = 1, \quad \psi(1) = 0, \tag{5.5}$$

$$\tilde{\psi}(z) = \frac{1}{(\lambda\tilde{\lambda})^{\rho}}\tilde{\psi}(\phi_0(z)), \qquad z \in \Omega(-\lambda^{-1}, a_1(\lambda)^{-1}), \quad \tilde{\psi}(0) = 1, \quad \tilde{\psi}(1) = 0. \tag{5.6}$$

The existence and properties of $\psi, \tilde{\psi}$ are well known from the literature. The function ψ is given by

$$\psi(z) = \frac{h(z)}{h(0)}, \qquad h(z) = \lim_{n\to\infty} \frac{1}{(\lambda\tilde{\lambda})^{n\tilde{\rho}}}(\tilde{\phi}_0^n(z) - 1),$$

and the limit converges uniformly on compact subsets of $\Omega(-\tilde{\lambda}^{-1}, \tilde{a}_1(\tilde{\lambda})^{-1})$, which is a basin of attraction of 1 for $\tilde{\phi}_0$. The function $\tilde{\psi}$ is defined in a similar way. We note that $\psi, \tilde{\psi}$ depend continuously on $\phi_0, \tilde{\phi}_0$. On $[-\tilde{\lambda}^{-1}, \tilde{a}_1(\tilde{\lambda})^{-1})$, ψ is strictly decreasing and, because $0 \leqslant \tilde{\phi}_0(-\tilde{\lambda}^{-1}) < 1$,

$$\psi(-\tilde{\lambda}^{-1}) = \frac{1}{(\lambda\tilde{\lambda})^{\tilde{\rho}}}\psi(\tilde{\phi}_0(-\tilde{\lambda}^{-1})) \leqslant \frac{1}{(\lambda\tilde{\lambda})^{\tilde{\rho}}}.$$

Similarly, we have

$$\tilde{\psi}(-\lambda^{-1}) = \frac{1}{(\lambda\tilde{\lambda})^{\rho}}\tilde{\psi}(\phi_0(-\lambda^{-1})) \leqslant \frac{1}{(\lambda\tilde{\lambda})^{\rho}}.$$

We now define new Herglotz functions $\phi, \tilde{\phi}$ by

$$\phi(z) = \frac{\psi(-\lambda z)^{\frac{1}{r}}}{\psi(-\lambda)^{\frac{1}{r}}}, \qquad z \in \Omega(-\lambda^{-1}, (\lambda\tilde{\lambda})^{-1}), \tag{5.7}$$

$$\tilde{\phi}(z) = \frac{\tilde{\psi}(-\tilde{\lambda} z)^{\frac{1}{r}}}{\tilde{\psi}(-\tilde{\lambda})^{\frac{1}{r}}}, \qquad z \in \Omega(-\tilde{\lambda}^{-1}, (\lambda\tilde{\lambda})^{-1}). \tag{5.8}$$

We have

$$\phi(-\lambda^{-1}) = 0, \quad \phi(1) = 1, \qquad \tilde{\phi}(-\tilde{\lambda}^{-1}) = 0, \qquad \tilde{\phi}(1) = 1,$$

$$\phi((\lambda\tilde{\lambda})^{-1}) = \frac{\psi(-\tilde{\lambda}^{-1})^{\frac{1}{r}}}{\psi(-\lambda)^{\frac{1}{r}}} < (\lambda\tilde{\lambda})^{-\frac{\tilde{\rho}}{r}} \leqslant (\lambda\tilde{\lambda})^{-1},$$

$$\tilde{\phi}((\lambda\tilde{\lambda})^{-1}) = \frac{\tilde{\psi}(-\lambda^{-1})^{\frac{1}{r}}}{\tilde{\psi}(-\tilde{\lambda})^{\frac{1}{r}}} < (\lambda\tilde{\lambda})^{-\frac{\rho}{r}} \leqslant (\lambda\tilde{\lambda})^{-1}.$$

Thus, the domain $\Omega(-\lambda^{-1}, (\lambda\tilde{\lambda})^{-1})$ is a basin of attraction of the fixed point 1 of ϕ, and the domain $\Omega(-\tilde{\lambda}^{-1}, (\lambda\tilde{\lambda})^{-1})$ is a basin of attraction of the fixed point 1 of $\tilde{\phi}$. The domains $\Omega(-\lambda^{-1}, (\lambda\tilde{\lambda})^{-1}), \Omega(-\tilde{\lambda}^{-1}, (\lambda\tilde{\lambda})^{-1})$ contain the domains $\Omega(-\lambda^{-1}, a_1(\lambda)^{-1})$, $\Omega(-\tilde{\lambda}^{-1}, \tilde{a}_1(\tilde{\lambda})^{-1})$ respectively, since $a_1(\lambda) \geqslant \lambda\tilde{\lambda}, \tilde{a}_1(\tilde{\lambda}) \geqslant \lambda\tilde{\lambda}$.

The fact that $\phi, \tilde{\phi}$ are Herglotz functions with $\phi(1) = 1, \tilde{\phi}(1) = 1, \phi((\lambda\tilde{\lambda})^{-1}) < (\lambda\tilde{\lambda})^{-1}, \tilde{\phi}((\lambda\tilde{\lambda})^{-1}) < (\lambda\tilde{\lambda})^{-1}$ implies that $\phi(a_1(\lambda)^{-1}) < a_1(\lambda)^{-1}$ and $\tilde{\phi}(\tilde{a}_1(\tilde{\lambda})^{-1}) < \tilde{a}_1(\tilde{\lambda})^{-1}$, so that ϕ maps the domain $\Omega(-\lambda^{-1}, a_1(\lambda)^{-1})$ into itself, and similarly $\tilde{\phi}$ maps the domain $\Omega(-\tilde{\lambda}^{-1}, \tilde{a}_1(\tilde{\lambda})^{-1})$ into itself.

5.1.2. Upper bounds for $\phi'(1), \tilde{\phi}'(1)$. We now obtain upper bounds for $\phi'(1)$ and $\tilde{\phi}'(1)$. From Schwarz's lemma we have

$$\phi'(1) \leqslant \frac{A'B'(A+B)}{AB(A'+B')} \tag{5.9}$$

with

$$A = 1 + \frac{1}{\lambda}, \qquad B = \frac{1}{\lambda\tilde{\lambda}} - 1, \qquad A' = 1, \qquad B' = \frac{1}{\lambda\tilde{\lambda}} - 1.$$

This gives

$$\phi'(1) \leqslant \frac{\lambda(1+\tilde{\lambda})}{1+\lambda} \leqslant \frac{\sqrt{\frac{\Lambda_+}{\gamma}} + \lambda\tilde{\lambda}}{\sqrt{\frac{\Lambda_+}{\gamma}} + 1}. \tag{5.10}$$

Similarly, we obtain

$$\tilde{\phi}'(1) \leqslant \frac{\tilde{\lambda}(1+\lambda)}{1+\tilde{\lambda}} \leqslant \frac{\sqrt{\gamma\Lambda_+} + \lambda\tilde{\lambda}}{\sqrt{\gamma\Lambda_+} + 1}. \tag{5.11}$$

Now, we have

$$\phi'(1) = -\frac{\lambda\psi'(-\lambda)}{r\psi(-\lambda)},$$

and from inequality (4.6), with $M = (\lambda\tilde{\lambda})^{-r}$, $m_1 = \tilde{\lambda}$, we obtain

$$\phi'(1) \leqslant \frac{\log(\lambda\tilde{\lambda})^{-1}}{4(1-\lambda\tilde{\lambda})}. \tag{5.12}$$

Similarly,

$$\tilde{\phi}'(1) \leqslant \frac{\log(\lambda\tilde{\lambda})^{-1}}{4(1-\lambda\tilde{\lambda})}. \tag{5.13}$$

Combining inequalities (5.10) and (5.12) we have

$$\phi'(1) \leqslant \min\left\{\frac{\sqrt{\frac{1}{\gamma}}+\lambda\tilde{\lambda}}{\sqrt{\frac{1}{\gamma}}+1}, \frac{\log(\lambda\tilde{\lambda})^{-1}}{4(1-\lambda\tilde{\lambda})}\right\} \leqslant \max_{0\leqslant x\leqslant 1}\min\left\{\frac{\sqrt{\frac{1}{\gamma}}+x}{\sqrt{\frac{1}{\gamma}}+1}, \frac{\log x^{-1}}{4(1-x)}\right\} = k < 1. \tag{5.14}$$

From inequalities (5.11) and (5.13) we also have

$$\tilde{\phi}'(1) \leqslant \max_{0\leqslant x\leqslant 1}\min\left\{\frac{\sqrt{\gamma}+x}{\sqrt{\gamma}+1}, \frac{\log x^{-1}}{4(1-x)}\right\} = \tilde{k} < 1. \tag{5.15}$$

We therefore choose $\Lambda_+ < 1$ such that $k_1 = \max(k, \tilde{k}) \leqslant \Lambda_+^r < 1$.

5.1.3. Lower bounds for $\phi'(1)$, $\tilde{\phi}'(1)$. We now consider the lower bounds. The lower bound in (4.5), with $m_2 = \tilde{a}_1(\tilde{\lambda}) < \tilde{a}_1(0) = \frac{\tilde{b}+\Lambda_+}{1+\tilde{b}}$ (see (5.4)), gives

$$\phi'(1) \geqslant \frac{\lambda(1-\tilde{a}_1(\tilde{\lambda}))}{r(1+\lambda)(1+\lambda\tilde{a}_1(\tilde{\lambda}))} \geqslant \frac{\lambda(1-\Lambda_+)}{r(1+b)(2+b+\tilde{b})} = \lambda K_1(r, \gamma, \Lambda_+). \tag{5.16}$$

Similarly, we have $\tilde{\phi}'(1) \geqslant \tilde{\lambda}\tilde{K}_1$, where $\tilde{K}_1 > 0$ depends on r, γ, Λ_+. Hence

$$\phi'(1)\tilde{\phi}'(1) \geqslant \lambda\tilde{\lambda}K_1\tilde{K}_1 = \Lambda K_1\tilde{K}_1 \geqslant \Lambda_-^2 K_1\tilde{K}_1, \tag{5.17}$$

since $\Lambda = \min\{\Lambda_0, \tilde{\Lambda}_0\} \geqslant \Lambda_0\tilde{\Lambda}_0 \geqslant \Lambda_-^2$. We now choose Λ_- such that

$$\Lambda_- \leqslant (K_1\tilde{K}_1)^{\frac{1}{2(r-1)}}, \tag{5.18}$$

and we then have $\phi'(1)\tilde{\phi}'(1) \geqslant \Lambda_-^{2r}$.

We now define the action of the operator $B(r, \gamma, \Lambda_+)$ on $(\Phi_0, \tilde{\Phi}_0)$ by

$$B(\Phi_0, \tilde{\Phi}_0) = (\Phi, \tilde{\Phi}), \tag{5.19}$$

where

$$\Phi = h_{b,\lambda}^{-1} \circ \phi \circ h_{b,\lambda}, \qquad \tilde{\Phi} = h_{\tilde{b},\tilde{\lambda}}^{-1} \circ \tilde{\phi} \circ h_{\tilde{b},\tilde{\lambda}}. \tag{5.20}$$

Our estimates show that if $\Lambda_+ \geqslant k_1^{\frac{1}{r}}$ and Λ_- satisfies (5.18), then

$$B(r, \gamma, \Lambda_+)\mathcal{Q}_1(r, \gamma, \Lambda_+) \subset \mathcal{Q}_1(r, \gamma, \Lambda_+).$$

The continuous map $B(r, \gamma, \Lambda_+)$ maps the compact convex non-empty set $\mathcal{Q}_1(r, \gamma, \Lambda_+)$ into itself. Therefore it has a fixed point there by the Schauder–Tikhonov theorem. If $(\Phi_0, \tilde{\Phi}_0) = (\Phi, \tilde{\Phi})$ is such a fixed point, the functions $\phi_0, \tilde{\phi}_0$ and $\phi, \tilde{\phi}$ constructed as above coincide respectively, and theorem 1 has been proved in this case with $a = \tilde{a} = \lambda\tilde{\lambda}$, $z_1 = \psi(-\lambda)^{-\frac{1}{r}}$, $\tilde{z}_1 = \tilde{\psi}(-\tilde{\lambda})^{-\frac{1}{r}}$.

5.2. The case $p \geqslant 2$

In this section $r > 1$ and $\gamma > 0$ are fixed real numbers, and $p \geqslant 2$ is a fixed integer. The real numbers Λ_- and Λ_+, with $0 < \Lambda_- < \Lambda_+ < 1$, are also fixed, but will be chosen later. Let $b = \sqrt{\frac{\Lambda_+}{\gamma}}$, $\tilde{b} = \sqrt{\gamma\Lambda_+}$, and note that $b\tilde{b} = \Lambda_+$.

We define the functions $a : [0, 1] \to [0, 1]$, $\tilde{a} : [0, 1] \to [0, 1]$ by

$$a(t) = \min\left\{\frac{(1+\sqrt{\gamma})\sqrt{t}}{\sqrt{\gamma}(1-\sqrt{t})}, \frac{1+\sqrt{t}}{2}\right\} = \begin{cases} \frac{(1+\sqrt{\gamma})\sqrt{t}}{\sqrt{\gamma}(1-\sqrt{t})}, & 0 \leqslant t \leqslant \Lambda_*, \\ \frac{1+\sqrt{t}}{2}, & \Lambda_* \leqslant t \leqslant 1, \end{cases} \tag{5.21}$$

$$\tilde{a}(t) = \min\left\{\frac{(1+\sqrt{\gamma})\sqrt{t}}{1-\sqrt{t}}, \frac{1+\sqrt{t}}{2}\right\} = \begin{cases} \frac{(1+\sqrt{\gamma})\sqrt{t}}{1-\sqrt{t}}, & 0 \leqslant t \leqslant \tilde{\Lambda}_*, \\ \frac{1+\sqrt{t}}{2}, & \tilde{\Lambda}_* \leqslant t \leqslant 1. \end{cases} \tag{5.22}$$

The functions $a, \tilde{a}$ are continuous and strictly increasing in [0, 1]. The numbers $\Lambda_*, \tilde{\Lambda}_*$, with $\Lambda_- < \Lambda_* < \Lambda_+$, $\Lambda_- < \tilde{\Lambda}_* < \Lambda_+$, satisfy $\frac{(1+\sqrt{\gamma})\sqrt{\Lambda_*}}{\sqrt{\gamma}(1-\sqrt{\Lambda_*})} = \frac{1+\sqrt{\Lambda_*}}{2}$, $\frac{(1+\sqrt{\gamma})\sqrt{\tilde{\Lambda}_*}}{1-\sqrt{\tilde{\Lambda}_*}} = \frac{1+\sqrt{\tilde{\Lambda}_*}}{2}$. Let $\Lambda_\star = \min\{\Lambda_*, \tilde{\Lambda}_*\}$.

We denote $\mathcal{Q}_2(r, \gamma, \Lambda_+)$ the space of all pairs of functions $(\Phi, \tilde{\Phi})$ with the following properties:

($\tilde{q}1$) $\Phi \in H(-b^{-1}, a(\Lambda_+)^{-1})$, $a(\Lambda_+) = \frac{1}{2}(1 + \sqrt{\Lambda_+})$, $\tilde{\Phi} \in H(-\tilde{b}^{-1}, \tilde{a}(\Lambda_+)^{-1})$, $\tilde{a}(\Lambda_+) = a(\Lambda_+)$,

($\tilde{q}2$) $\Phi(-b^{-1}, a(\Lambda_+)^{-1}) \subseteq (0, a(\Lambda_+)^{-1})$, $\tilde{\Phi}(-\tilde{b}^{-1}, \tilde{a}(\Lambda_+)^{-1}) \subseteq (0, \tilde{a}(\Lambda_+)^{-1})$,

($\tilde{q}3$) $\Phi(1) = 1$, $\tilde{\Phi}(1) = 1$, $\Phi'(1) \leqslant \Lambda_+^r$, $\tilde{\Phi}'(1) \leqslant \Lambda_+^r$, $\Phi'(1)\tilde{\Phi}'(1) \geqslant \Lambda_-^{2r}$.

We shall define a continuous operator $B(r, p, \gamma, \Lambda_+)$ by describing its action on an arbitrary $(\Phi_0, \tilde{\Phi}_0) \in \mathcal{Q}_2(r, \gamma, \Lambda_+)$. As in the previous section, we first define functions $\phi_0, \tilde{\phi}_0$, and their linearizers $\psi, \tilde{\psi}$, respectively. Then, we proceed to define the associated functions $\phi, \tilde{\phi}$, and obtain bounds for $\phi'(1), \tilde{\phi}'(1)$.

5.2.1. The functions $\phi, \tilde{\phi}$. Given $(\Phi_0, \tilde{\Phi}_0) \in \mathcal{Q}_2(r, \gamma, \Lambda_+)$, we denote $\Lambda_0 = \Phi_0'(1)^{\frac{1}{r}}$, $\tilde{\Lambda}_0 = \tilde{\Phi}_0'(1)^{\frac{1}{r}}$. Then $\Lambda_0 \leqslant \Lambda_+$, $\tilde{\Lambda}_0 \leqslant \Lambda_+$ and $\Lambda_0\tilde{\Lambda}_0 \geqslant \Lambda_-^2$. We set $\Lambda = \min\{\Lambda_0, \tilde{\Lambda}_0\}$, $\lambda = \sqrt{\frac{\Lambda}{\gamma}}$, $\tilde{\lambda} = \sqrt{\gamma\Lambda}$, $\rho = \frac{r\log\Lambda_0}{\log\Lambda} \leqslant r$ and $\tilde{\rho} = \frac{r\log\tilde{\Lambda}_0}{\log\Lambda} \leqslant r$. Note that $0 < \lambda \leqslant b$, $0 < \tilde{\lambda} \leqslant \tilde{b}$. We define functions $\phi_0, \tilde{\phi}_0$ by

$$\phi_0 = h_{b,\lambda} \circ \Phi_0 \circ h_{b,\lambda}^{-1}, \qquad \tilde{\phi}_0 = h_{\tilde{b},\tilde{\lambda}} \circ \tilde{\Phi}_0 \circ h_{\tilde{b},\tilde{\lambda}}^{-1}, \tag{5.23}$$

where the function h was defined in (5.1). Thus, $h_{b,\lambda}$ maps $\Omega(-b^{-1}, a(\Lambda_+)^{-1})$ onto $\Omega(-\lambda^{-1}, a_2(\lambda)^{-1})$ where

$$a_2(\lambda)^{-1} = h_{b,\lambda}(a(\Lambda_+)^{-1}),$$

$$a(\Lambda) \leqslant a(\Lambda_+) \leqslant a_2(\lambda) = \frac{2b + 1 + \sqrt{\Lambda_+} - \lambda(1 - \sqrt{\Lambda_+})}{2(1+b)} < a_2(0) = \frac{2b + 1 + \sqrt{\Lambda_+}}{2(1+b)} < 1. \tag{5.24}$$

Similarly, $h_{\tilde{b},\tilde{\lambda}}$ maps $\Omega(-\tilde{b}^{-1}, \tilde{a}(\Lambda_+)^{-1})$ onto $\Omega(-\tilde{\lambda}^{-1}, \tilde{a}_2(\tilde{\lambda})^{-1})$, where

$$\tilde{a}_2(\tilde{\lambda})^{-1} = h_{\tilde{b},\tilde{\lambda}}(\tilde{a}(\Lambda_+)^{-1}),$$

$$\tilde{a}(\Lambda) \leqslant \tilde{a}(\Lambda_+) \leqslant \tilde{a}_2(\tilde{\lambda}) = \frac{2\tilde{b} + 1 + \sqrt{\Lambda_+} - \tilde{\lambda}(1 - \sqrt{\Lambda_+})}{2(1+\tilde{b})} < \tilde{a}_2(0) = \frac{2\tilde{b} + 1 + \sqrt{\Lambda_+}}{2(1+\tilde{b})} < 1. \tag{5.25}$$

The functions $\phi_0, \tilde{\phi}_0$ possess the same properties as in the previous section, but with $a_1(\lambda), \tilde{a}_1(\tilde{\lambda})$ now replaced by $a_2(\lambda), \tilde{a}_2(\tilde{\lambda})$, respectively.

As before, we denote $\psi, \tilde{\psi}$ the linearizers of $\phi_0, \tilde{\phi}_0$ respectively, normalized by the condition $\psi(0) = \tilde{\psi}(0) = 1$. Then,

$$\psi(z) = \frac{1}{(\lambda\tilde{\lambda})^{\tilde{\rho}}} \psi(\tilde{\phi}_0(z)), \qquad z \in \Omega(-\tilde{\lambda}^{-1}, \tilde{a}_2(\tilde{\lambda})^{-1}), \quad \psi(0) = 1, \quad \psi(1) = 0, \tag{5.26}$$

$$\tilde{\psi}(z) = \frac{1}{(\lambda\tilde{\lambda})^{\rho}} \tilde{\psi}(\phi_0(z)), \qquad z \in \Omega(-\lambda^{-1}, a_2(\lambda)^{-1}), \quad \tilde{\psi}(0) = 1, \quad \tilde{\psi}(1) = 0. \tag{5.27}$$

ψ and $\tilde{\psi}$ are anti-Herglotz functions, analytic in the domains $\Omega(-\tilde{\lambda}^{-1}, \tilde{a}_2(\tilde{\lambda})^{-1})$, $\Omega(-\lambda^{-1}, a_2(\lambda)^{-1})$ respectively, and they satisfy the following inequalities:

$$\psi(-\tilde{\lambda}^{-1}) = \frac{1}{(\lambda\tilde{\lambda})^{\tilde{\rho}}} \psi(\tilde{\phi}_0(-\tilde{\lambda}^{-1})) \leqslant \frac{1}{(\lambda\tilde{\lambda})^{\tilde{\rho}}},$$

$$\tilde{\psi}(-\lambda^{-1}) = \frac{1}{(\lambda\tilde{\lambda})^{\rho}} \tilde{\psi}(\phi_0(-\lambda^{-1})) \leqslant \frac{1}{(\lambda\tilde{\lambda})^{\rho}}.$$

We define Herglotz functions $v, \tilde{v}$ by

$$v(z) = \psi(-z)^{\frac{1}{r}}, \qquad z \in \Omega(-1, \tilde{\lambda}^{-1}), \tag{5.28}$$

$$\tilde{v}(z) = \tilde{\psi}(-z)^{\frac{1}{r}}, \qquad z \in \Omega(-1, \lambda^{-1}). \tag{5.29}$$

They satisfy

$$v(-1) = 0, \qquad v(0) = 1, \qquad \tilde{v}(-1) = 0, \qquad \tilde{v}(0) = 1,$$

$$v(\tilde{\lambda}^{-1}) \leqslant (\lambda\tilde{\lambda})^{-\frac{\tilde{\rho}}{r}} \leqslant (\lambda\tilde{\lambda})^{-1}, \qquad \tilde{v}(\lambda^{-1}) \leqslant (\lambda\tilde{\lambda})^{-\frac{\rho}{r}} \leqslant (\lambda\tilde{\lambda})^{-1}.$$

We now show that there exist unique $z_1 \in (0, 1), \tilde{z}_1 \in (0, 1)$ such that

$$(z_1\tilde{\lambda}^{-1}v)^p(\lambda) = \tilde{\lambda}^{-1}, \tag{5.30}$$

$$(\tilde{z}_1\lambda^{-1}\tilde{v})^p(\tilde{\lambda}) = \lambda^{-1}. \tag{5.31}$$

Following the arguments in [E2] closely, we prove (5.30); (5.31) follows in a similar way. For real $s \geqslant 0$ let $x_0(s) = \lambda, x_1(s) = s\tilde{\lambda}^{-1}v(\lambda)$. The function $s \to x_1(s)$ is strictly increasing on $\mathbb{R}_+$ and takes the values λ at $s_* = \lambda\tilde{\lambda}/v(\lambda)$ and $\tilde{\lambda}^{-1}$ at $s_1 = 1/v(\lambda)$. Note that, since $v(\lambda) > v(0) = 1, s_* < \lambda\tilde{\lambda}$ and $s_1 < 1$. Also, $s_* \geqslant (\lambda\tilde{\lambda})/v(\tilde{\lambda}^{-1}) \geqslant (\lambda\tilde{\lambda})/(\lambda\tilde{\lambda})^{-1} = (\lambda\tilde{\lambda})^2$. By induction we can construct a strictly decreasing infinite sequence $s_1 > \cdots > s_j > \cdots > s_*$ such that, for $j \geqslant 2, x_j(s) = (s\tilde{\lambda}^{-1}v)^j(\lambda)$ is continuous and strictly increasing on $[s_*, s_{j-1}], x_0(s) < \cdots < x_j(s)$ in $(s_*, s_{j-1}], x_j(s_*) = \lambda$, and $x_j(s_j) = \tilde{\lambda}^{-1}$. Indeed, $x_{j+1}(s) = s\tilde{\lambda}^{-1}v(x_j(s))$ is defined, continuous and strictly increasing on $[s_*, s_j]$, and $x_{j+1}(s) > x_j(s)$ for all $s \in (s_*, s_j]$. Since $x_{j+1}(s_j) > x_j(s_j) = \tilde{\lambda}^{-1}$ and $x_{j+1}(s_*) = \lambda, s_{j+1}$

exists in (s_*, s_j). In particular, $x_p(s_{p-1}) > \tilde{\lambda}^{-1}$ and therefore there is a unique $z_1 \in (s_*, s_{p-1})$ such that $x_p(z_1) = \tilde{\lambda}^{-1}$. Note that $z_1 < 1$, since $z_1 < s_{p-1} < s_1$.

We also note that for $s \in (s_*, s_{p-1})$, there exists a unique $x_{-1}(s)$, with $-1 < x_{-1}(s) < \lambda$, such that $s\tilde{\lambda}^{-1}v(x_{-1}(s)) = \lambda$. The function $s_*\tilde{\lambda}^{-1}v(z)$ maps $\Omega(-1, \tilde{\lambda}^{-1})$ into $\Omega(0, \tilde{\lambda}^{-1}/v(\lambda))$, so that it has a unique and attractive fixed point at λ by Schwarz's lemma. Thus $s_*\tilde{\lambda}^{-1}v(x) \geqslant x$ for all $x \in [-1, \lambda]$, and when $s > s_*$, $s\tilde{\lambda}^{-1}v(x) > x$ for all $x \in [-1, \lambda]$. Since this includes $[x_{-1}(s), x_0(s)]$, it follows that $s\tilde{\lambda}^{-1}v(x) > x$ for all $x \in [-1, x_p(s)]$, for all $s \in (s_*, z_1]$. To see this, note that since the function $s\tilde{\lambda}^{-1}v$ is continuous and strictly increasing, the intermediate value theorem implies that for any $x^{(j)} \in [x_{j-1}(s), x_j(s)]$, $1 \leqslant j \leqslant p$, there exists $x^{(j-1)} \in [x_{j-2}(s), x_{j-1}(s)]$ such that $s\tilde{\lambda}^{-1}v(x^{(j-1)}) = x^{(j)}$. It follows that for any $x \in [-1, x_p(s)]$, $s \in (s_*, z_1]$, there exists $x^{(0)} \in [-1, x_0(s)]$ such that $(s\tilde{\lambda}^{-1}v)^m(x^{(0)}) = x$, for some integer m, $0 \leqslant m \leqslant p$. Then $s\tilde{\lambda}^{-1}v(x^{(0)}) > x^{(0)} \Rightarrow s\tilde{\lambda}^{-1}v(s\tilde{\lambda}^{-1}v(x^{(0)})) > s\tilde{\lambda}^{-1}v(x^{(0)}) \Rightarrow \cdots \Rightarrow s\tilde{\lambda}^{-1}v((s\tilde{\lambda}^{-1}v)^m(x^{(0)})) > (s\tilde{\lambda}^{-1}v)^m(x^{(0)})$, i.e. $s\tilde{\lambda}^{-1}v(x) > x$ as stated.

For $0 \leqslant j \leqslant p+1$, we denote $\zeta_j = (z_1\tilde{\lambda}^{-1}v)^j(\lambda)$. Then, we have

$$\lambda = \zeta_0 < \zeta_1 < \cdots < \zeta_p = \tilde{\lambda}^{-1} < \zeta_{p+1} = z_1\tilde{\lambda}^{-1}v(\tilde{\lambda}^{-1}).$$

Since $v(\tilde{\lambda}^{-1}) \leqslant (\lambda\tilde{\lambda})^{-1}$

$$z_1 > \lambda\tilde{\lambda}, \tag{5.32}$$

and since $v(\lambda) > v(0) = 1$

$$\zeta_1 > \frac{z_1}{\tilde{\lambda}}. \tag{5.33}$$

Similarly we denote $\tilde{\zeta}_j = (\tilde{z}_1\lambda^{-1}\tilde{v})^j(\tilde{\lambda})$, and obtain

$$\tilde{z}_1 > \lambda\tilde{\lambda}, \qquad \tilde{\zeta}_1 > \frac{\tilde{z}_1}{\lambda}. \tag{5.34}$$

We have seen above that

$$z_1\tilde{\lambda}^{-1}v(x) > x, \qquad x \in [-1, \tilde{\lambda}^{-1}]. \tag{5.35}$$

Setting $x = \sqrt{\lambda\tilde{\lambda}^{-1}}$ gives $z_1\tilde{\lambda}^{-1} > \sqrt{\lambda\tilde{\lambda}^{-1}}/v(\sqrt{\lambda\tilde{\lambda}^{-1}})$, and using (4.4),

$$z_1\tilde{\lambda}^{-1} > \sqrt{\frac{\lambda}{\tilde{\lambda}}}\left(\frac{1-\sqrt{\lambda\tilde{\lambda}}}{1+\sqrt{\frac{\lambda}{\tilde{\lambda}}}}\right)^{\frac{1}{r}} > \sqrt{\frac{\lambda}{\tilde{\lambda}}}\left(\frac{1-\sqrt{\lambda\tilde{\lambda}}}{1+\sqrt{\frac{\lambda}{\tilde{\lambda}}}}\right) = \frac{1-\sqrt{\Lambda}}{1+\sqrt{\gamma}}.$$

Hence, in view of (5.33),

$$\frac{\zeta_1}{\lambda} > \frac{\sqrt{\gamma}(1-\sqrt{\Lambda})}{(1+\sqrt{\gamma})\sqrt{\Lambda}}. \tag{5.36}$$

Similarly, we have $\tilde{z}_1\lambda^{-1}\tilde{v}(x) > x$, for $x \in [-1, \lambda^{-1}]$. Setting $x = \sqrt{\tilde{\lambda}\lambda^{-1}}$ we obtain, in view of (5.34),

$$\frac{\tilde{\zeta}_1}{\tilde{\lambda}} > \frac{1-\sqrt{\Lambda}}{(1+\sqrt{\gamma})\sqrt{\Lambda}}. \tag{5.37}$$

We now define new Herglotz functions $\phi, \tilde{\phi}$ by

$$\phi(z) = \tilde{\lambda}(z_1\tilde{\lambda}^{-1}v)^p(\lambda z), \qquad z \in \Omega(-\lambda^{-1}, \zeta_1\lambda^{-1}), \tag{5.38}$$

$$\tilde{\phi}(z) = \lambda(\tilde{z}_1\lambda^{-1}\tilde{v})^p(\tilde{\lambda}z), \qquad z \in \Omega(-\tilde{\lambda}^{-1}, \tilde{\zeta}_1\tilde{\lambda}^{-1}). \tag{5.39}$$

We have

$$\phi(-\lambda^{-1}) = \tilde{\lambda}(z_1\tilde{\lambda}^{-1}v)^{p-1}(0) \geqslant z_1 > \lambda\tilde{\lambda}, \qquad \phi(1) = 1,$$
$$\phi(\zeta_1\lambda^{-1}) = z_1 v(\tilde{\lambda}^{-1}) < z_1(\lambda\tilde{\lambda})^{-1} < \zeta_1\lambda^{-1}.$$

We also have

$$\tilde{\phi}(-\tilde{\lambda}^{-1}) \geqslant \tilde{z}_1 > \lambda\tilde{\lambda}, \qquad \tilde{\phi}(1) = 1, \qquad \tilde{\phi}(\tilde{\zeta}_1\tilde{\lambda}^{-1}) < \tilde{\zeta}_1\tilde{\lambda}^{-1}.$$

Therefore, the domain $\Omega(-\lambda^{-1}, \zeta_1\lambda^{-1})$ is a basin of attraction of the fixed point 1 of ϕ, and the domain $\Omega(-\tilde{\lambda}^{-1}, \tilde{\zeta}_1\tilde{\lambda}^{-1})$ is a basin of attraction of the fixed point 1 of $\tilde{\phi}$. The following lemma implies that the domains $\Omega(-\lambda^{-1}, \zeta_1\lambda^{-1}), \Omega(-\tilde{\lambda}^{-1}, \tilde{\zeta}_1\tilde{\lambda}^{-1})$ contain the domains $\Omega(-\lambda^{-1}, a_2(\lambda)^{-1}), \Omega(-\tilde{\lambda}^{-1}, \tilde{a}_2(\tilde{\lambda})^{-1})$, respectively.

Lemma 2. *The inequality*

$$z_1\tilde{\lambda}^{-1}v(z) \geqslant \frac{(1-c\tilde{\lambda})cz + (c-z)\lambda}{c(1-\tilde{\lambda}z)} \tag{5.40}$$

holds for all $z \in [0, c]$, $c \in [\lambda, \tilde{\lambda}^{-1})$.

Proof. The result follows from lemma 1 of section 4, with $f = z_1\tilde{\lambda}^{-1}v$, and $a = 0, b = c, c \in [\lambda, \tilde{\lambda}^{-1})$, and $B = \tilde{\lambda}^{-1}$. This function satisfies $f(0) = z_1\tilde{\lambda}^{-1} \geqslant \lambda$ by (5.32), and $f(c) \geqslant c$ by (5.35). □

For $z = \lambda$, and choosing $c = \sqrt{\lambda\tilde{\lambda}^{-1}}$, we obtain $\zeta_1/\lambda \geqslant 2/(1+\sqrt{\lambda\tilde{\lambda}})$. Since we also have the lower bound (5.36),

$$\frac{\zeta_1}{\lambda} \geqslant \max\left\{\frac{2}{1+\sqrt{\Lambda}}, \frac{\sqrt{\gamma}(1-\sqrt{\Lambda})}{(1+\sqrt{\gamma})\sqrt{\Lambda}}\right\} = \frac{1}{a(\Lambda)}. \tag{5.41}$$

In a similar way we find that $\tilde{\zeta}_1/\tilde{\lambda} \geqslant 2/(1+\sqrt{\lambda\tilde{\lambda}})$, and together with the lower bound (5.37) we have

$$\frac{\tilde{\zeta}_1}{\tilde{\lambda}} \geqslant \max\left\{\frac{2}{1+\sqrt{\Lambda}}, \frac{1-\sqrt{\Lambda}}{(1+\sqrt{\gamma})\sqrt{\Lambda}}\right\} = \frac{1}{\tilde{a}(\Lambda)}. \tag{5.42}$$

The above inequalities justify our definitions of the functions $a, \tilde{a}$ in (5.21) and (5.22), respectively. In view of (5.24) and (5.25), we conclude that the domains $\Omega(-\lambda^{-1}, \zeta_1\lambda^{-1}), \Omega(-\tilde{\lambda}^{-1}, \tilde{\zeta}_1\tilde{\lambda}^{-1})$, where $\phi, \tilde{\phi}$ are analytic and which they map into themselves, respectively, contain the domains of analyticity $\Omega(-\lambda^{-1}, a_2(\lambda)^{-1}), \Omega(-\tilde{\lambda}^{-1}, \tilde{a}_2(\tilde{\lambda})^{-1})$ of $\phi_0, \tilde{\phi}_0$, respectively.

5.2.2. Upper bounds for $\phi'(1), \tilde{\phi}'(1)$. We now use Schwarz's lemma to obtain upper bounds for $\phi'(1)$ and $\tilde{\phi}'(1)$. Using inequality (5.9) with

$$A = 1 + \frac{1}{\lambda}, \qquad B = B' = \frac{1}{a(\Lambda)} - 1, \qquad A' = 1 - \lambda\tilde{\lambda}$$

we obtain

$$\phi'(1) \leqslant \frac{(1-\lambda\tilde{\lambda})(1+\sqrt{\lambda\tilde{\lambda}}+2\lambda)}{(1+\lambda)(2-\lambda\tilde{\lambda}-(\lambda\tilde{\lambda})^{\frac{3}{2}})}. \tag{5.43}$$

Writing $X = \lambda\tilde{\lambda}$, we have $\lambda = \sqrt{\frac{X}{\gamma}}$ and

$$\phi'(1) \leqslant \frac{(1-X)\left(1+\sqrt{X}+2\sqrt{\frac{X}{\gamma}}\right)}{\left(1+\sqrt{\frac{X}{\gamma}}\right)(2-X-X^{\frac{3}{2}})} = f_1(X).$$

We consider $f_1(X), 0 \leqslant X \leqslant 1$. By calculation, we find

$$1 - f_1(X) = \frac{X^{\frac{3}{2}} + \sqrt{\gamma}}{(2 + 2\sqrt{X} + X)(\sqrt{X} + \sqrt{\gamma})} \geqslant \frac{\sqrt{\gamma}}{5(1 + \sqrt{\gamma})}.$$

We therefore have the upper bound

$$\phi'(1) \leqslant 1 - \frac{\sqrt{\gamma}}{5(1 + \sqrt{\gamma})} = f_2(\gamma). \tag{5.44}$$

Similarly, we have

$$\tilde{\phi}'(1) \leqslant 1 - \frac{1}{5(1 + \sqrt{\gamma})} = \tilde{f}_2(\gamma). \tag{5.45}$$

Let $k_2 = \max\{f_2(\gamma), \tilde{f}_2(\gamma)\}$. We choose Λ_+ such that

$$k_2^{\frac{1}{r}} \leqslant \Lambda_+ < 1. \tag{5.46}$$

We then have $\phi'(1) \leqslant \Lambda_+^r$ and $\tilde{\phi}'(1) \leqslant \Lambda_+^r$. It is easy to see that Λ_+ must satisfy $0.8 < \Lambda_+ < 1$.

5.2.3. Lower bounds for $\phi'(1), \tilde{\phi}'(1)$. We now obtain lower bounds for $\phi'(1), \tilde{\phi}'(1)$ as follows. We have

$$\begin{aligned}\phi'(1) &= \lambda\tilde{\lambda} \prod_{j=0}^{p-1} z_1 \tilde{\lambda}^{-1} v'(\zeta_j) = \lambda\tilde{\lambda} \prod_{j=0}^{p-1} \frac{\zeta_{j+1} v'(\zeta_j)}{v(\zeta_j)} \\ &= \prod_{j=0}^{p-1} \frac{\zeta_j v'(\zeta_j)}{v(\zeta_j)} = \prod_{j=0}^{p-1} \frac{-\zeta_j \psi'(-\zeta_j)}{r\psi(-\zeta_j)},\end{aligned} \tag{5.47}$$

where we have used $\zeta_0 = \lambda\tilde{\lambda}\zeta_p$. The lower bound in (4.5) gives

$$-\frac{\zeta\psi'(-\zeta)}{\psi(-\zeta)} \geqslant \frac{\zeta(1 - \tilde{c})}{(1 + \zeta)(1 + \tilde{c}\zeta)} = g(\zeta), \qquad \zeta \in [0, \tilde{\lambda}^{-1}], \tag{5.48}$$

with $\tilde{c} = \tilde{a}_2(\tilde{\lambda})$, where $\tilde{a}_2(\tilde{\lambda})$ was defined in (5.25) and satisfies

$$\sqrt{\Lambda} < \tilde{a}(\Lambda) \leqslant \tilde{a}(\Lambda_+) \leqslant \tilde{a}_2(\tilde{\lambda}) < \tilde{a}_2(0) = \frac{2\tilde{b} + 1 + \sqrt{\Lambda_+}}{2(\tilde{b} + 1)} < 1. \tag{5.49}$$

We suppose at the moment that (5.48) holds with some number $\tilde{c}$ satisfying $\tilde{a}(\Lambda) \leqslant \tilde{c} < 1$. For $0 < j \leqslant p$, $\tilde{\lambda}^{-1} = \zeta_p \geqslant \zeta_j \geqslant \zeta_1 \geqslant \lambda/a(\Lambda) \geqslant \lambda/\tilde{c}$, so that $\zeta_j \in [\lambda/\tilde{c}, 1/\tilde{\lambda}]$. It is easy to check that on this interval, the function g attains its minimum value at one of its endpoints. By calculation we find that

$$g\left(\frac{\lambda}{\tilde{c}}\right) - g\left(\frac{1}{\tilde{\lambda}}\right) = \frac{(\lambda - \tilde{\lambda})(1 - \tilde{c})(\tilde{c} - \Lambda)}{(\lambda + \tilde{c})(1 + \lambda)(\tilde{\lambda} + \tilde{c})(1 + \tilde{\lambda})}.$$

If $\lambda < \tilde{\lambda}$ ($\gamma > 1$), then g attains its minimum value at $\zeta = \lambda/\tilde{c}$ and we have the lower bound

$$\phi'(1) \geqslant \lambda^p \left(\frac{(1 - \tilde{c})}{r(1 + \lambda)(1 + \tilde{c}\lambda)}\right)\left(\frac{(1 - \tilde{c})}{r(\lambda + \tilde{c})(1 + \lambda)}\right)^{p-1} \tag{5.50}$$

$$\geqslant \lambda^p \left(\frac{1 - \tilde{c}}{r(1 + \lambda)^2}\right)^p. \tag{5.51}$$

Setting now $\tilde{c} = \tilde{a}_2(\tilde{\lambda})$, and using inequalities (5.49) and $\lambda \leqslant b$, we obtain

$$\phi'(1) \geqslant \lambda^p \left(\frac{1 - \sqrt{\Lambda_+}}{2r(1 + \tilde{b})(1 + b)^2}\right)^p = \lambda^p K_2(r, p, \gamma, \Lambda_+). \tag{5.52}$$

Otherwise, if $\lambda > \tilde{\lambda}$ $(0 < \gamma < 1)$, then g attains its minimum value at $\zeta = \tilde{\lambda}^{-1}$, and we have the bound

$$\phi'(1) \geqslant \lambda\tilde{\lambda}^{p-1}\left(\frac{1-\sqrt{\Lambda_+}}{2r}\right)^p \frac{1}{(1+b)^2(1+\tilde{b})^{3p-2}} = \lambda\tilde{\lambda}^{p-1}K_2'(r,p,\gamma,\Lambda_+). \quad (5.53)$$

Note that when $\lambda = \tilde{\lambda}$ $(\gamma = 1)$, both (5.52) and (5.53) hold.

Similarly, one can verify that if $\lambda < \tilde{\lambda}$ then $\tilde{\phi}'(1) \geqslant \tilde{\lambda}\lambda^{p-1}\tilde{K}_2(r,p,\gamma,\Lambda_+)$, and if $\lambda > \tilde{\lambda}$ then $\tilde{\phi}'(1) \geqslant \tilde{\lambda}^p\tilde{K}_2'(r,p,\gamma,\Lambda_+)$, for some functions $\tilde{K}_2, \tilde{K}_2'$ satisfying $0 < \tilde{K}_2 < 1, 0 < \tilde{K}_2' < 1$.

Suppose that $\lambda \leqslant \tilde{\lambda}$. Then we have

$$\phi'(1)\tilde{\phi}'(1) \geqslant \tilde{\lambda}\lambda^{2p-1}K_2(r,p,\gamma,\Lambda_+)\tilde{K}_2(r,p,\gamma,\Lambda_+) = (\lambda\tilde{\lambda})^p\gamma^{1-p}K_2\tilde{K}_2 \geqslant \Lambda_-^{2p}\gamma^{1-p}K_2\tilde{K}_2, \quad (5.54)$$

since $\lambda\tilde{\lambda} = \Lambda = \min\{\Lambda_0, \tilde{\Lambda}_0\} \geqslant \Lambda_0\tilde{\Lambda}_0 \geqslant \Lambda_-^2$. We now choose Λ_- such that

$$\Lambda_- \leqslant (\gamma^{1-p}K_2\tilde{K}_2)^{\frac{1}{2(r-p)}}. \quad (5.55)$$

Then, provided $r > p$, it follows that $\phi'(1)\tilde{\phi}'(1) \geqslant \Lambda_-^{2r}$. The case $\lambda > \tilde{\lambda}$ is similar. We note that the condition $r > p$ is just a limitation of the present method, and is the result of the fact that $a_2(\lambda)$ and $\tilde{a}_2(\tilde{\lambda})$ do not tend to 0 as λ and $\tilde{\lambda}$ respectively tend to 0. However, as we shall see below, in the case of fixed points, our estimates can be improved.

We now define the operator $B(r,p,\gamma,\Lambda_+)$ by

$$B(\Phi_0, \tilde{\Phi}_0) = (\Phi, \tilde{\Phi}), \quad (5.56)$$

where

$$\Phi = h_{b,\lambda}^{-1} \circ \phi \circ h_{b,\lambda}, \qquad \tilde{\Phi} = h_{\tilde{b},\tilde{\lambda}}^{-1} \circ \tilde{\phi} \circ h_{\tilde{b},\tilde{\lambda}}. \quad (5.57)$$

Our estimates show that if Λ_+ is chosen so as to satisfy (5.46), Λ_- is chosen so as to satisfy (5.55) (in the $\lambda \leqslant \tilde{\lambda}$ case, and a similar inequality when $\lambda > \tilde{\lambda}$), and $r > p$, then

$$B(r,p,\gamma,\Lambda_+)\mathcal{Q}_2(r,\gamma,\Lambda_+) \subset \mathcal{Q}_2(r,\gamma,\Lambda_+).$$

The continuous operator $B(r,p,\gamma,\Lambda_+)$ maps the compact convex non-empty set $\mathcal{Q}_2(r,\gamma,\Lambda_+)$ into itself and therefore, as in the preceding section, by the Schauder–Tikhonov theorem we obtain a fixed point which provides a solution to our problem.

In the case of fixed points, the lower bound on Λ can be improved. The functions $\psi, \tilde{\phi}$ are then analytic in $\Omega(-\tilde{\lambda}^{-1}, \tilde{a}(\Lambda)^{-1})$, *and* the functions $\tilde{\psi}, \phi$ are analytic in $\Omega(-\lambda^{-1}, a(\Lambda)^{-1})$. Thus, the bound (5.48) now holds with $\tilde{c}$ replaced by $\tilde{a}(\Lambda)$, instead of $\tilde{a}_2(\tilde{\lambda})$. Assume that $\Lambda \leqslant \Lambda_\star$, and $\lambda < \tilde{\lambda}$. The case $\lambda > \tilde{\lambda}$ is similar. We can then set $\tilde{c} = \tilde{a}(\Lambda) = (1+\sqrt{\gamma})\sqrt{\Lambda}/(1-\sqrt{\Lambda})$ in (5.50), and after some calculations we obtain

$$\phi'(1) \geqslant \lambda\left(\frac{1-(\sqrt{\gamma}+2)\sqrt{\Lambda_\star}}{r(1+b)}\right)^p \frac{(1+\sqrt{\gamma}(1+\sqrt{\gamma}))^{1-p}}{(1+b(1+\sqrt{\gamma}))} = \lambda K_3(r,p,\gamma,\Lambda_\star). \quad (5.58)$$

Thus, for fixed points we have

$$\Lambda_0 = \phi'(1)^{\frac{1}{r}} \geqslant \left(\frac{\Lambda}{\gamma}\right)^{\frac{1}{2r}} K_3^{\frac{1}{r}}. \quad (5.59)$$

Similarly, in the case of fixed points, we obtain

$$\tilde{\Lambda}_0 = \tilde{\phi}'(1)^{\frac{1}{r}} \geqslant (\gamma\Lambda)^{\frac{1}{2r}}\tilde{K}_3^{\frac{1}{r}}, \quad (5.60)$$

for some function $\tilde{K}_3 = \tilde{K}_3(r, p, \gamma, \Lambda_\star)$. From (5.59) and (5.60) we see that for fixed points we have $\Lambda_0\tilde{\Lambda}_0 \geqslant \Lambda^{\frac{1}{r}}(K_3\tilde{K}_3)^{\frac{1}{r}}$, and since $\Lambda = \min\{\Lambda_0, \tilde{\Lambda}_0\} \geqslant \Lambda_0\tilde{\Lambda}_0$, $\Lambda \geqslant (K_3\tilde{K}_3)^{\frac{1}{r-1}}$. Therefore, the lower bound

$$\Lambda \geqslant \min\left\{\Lambda_\star, (K_3\tilde{K}_3)^{\frac{1}{r-1}}\right\} \tag{5.61}$$

holds for all fixed points.

This fact suggests the use of another operator instead of B, and this will be done in the following subsection.

5.3. The operator N

In this subsection we define an operator N on the space $\tilde{\mathcal{Q}}_2(r, \gamma, \Lambda_+)$, where $\tilde{\mathcal{Q}}_2(r, \gamma, \Lambda_+)$ has the same properties as the space $\mathcal{Q}_2(r, \gamma, \Lambda_+)$ in the preceding subsection, with the only exception that $\Phi'(1)\tilde{\Phi}'(1) > 0$ (property $(\tilde{q}_3)$). This new operator is a 'truncated version' of B (which is analytic on $\mathcal{Q}_2(r, \gamma, \Lambda_+)$), and depends on an additional real parameter $\Lambda_1 \in (0, 1/2)$. It is only continuous, but it maps $\tilde{\mathcal{Q}}_2(r, \gamma, \Lambda_+)$ into a compact set. It will be shown that any fixed point of N is a fixed point of B.

The notation is the same as in the previous subsection unless explicitly mentioned. In particular, $r > 1$ and $\gamma > 0$ are fixed real numbers, and Λ_+ is chosen as in (5.46).

5.3.1. Definition of N. We define $N(r, p, \gamma, \Lambda_1)$ by its action on an arbitrary element $(\Phi_0, \tilde{\Phi}_0)$ of $\tilde{\mathcal{Q}}_2(r, \gamma, \Lambda_+)$. We denote $\Lambda_0 = \Phi_0'(1)^{\frac{1}{r}}$, $\tilde{\Lambda}_0 = \tilde{\Phi}_0'(1)^{\frac{1}{r}}$. If $\min\{\Lambda_0, \tilde{\Lambda}_0\} \geqslant \Lambda_1$, then we set $\Lambda = \min\{\Lambda_0, \tilde{\Lambda}_0\}$ as before, and define

$$N(\Phi_0, \tilde{\Phi}_0) = B(\Phi_0, \tilde{\Phi}_0), \qquad \min\{\Lambda_0, \tilde{\Lambda}_0\} \geqslant \Lambda_1. \tag{5.62}$$

If $\min\{\Lambda_0, \tilde{\Lambda}_0\} < \Lambda_1$, then we set $\Lambda = \Lambda_1$. We define $\lambda = \lambda_1 = \sqrt{\frac{\Lambda_1}{\gamma}}$, $\tilde{\lambda} = \tilde{\lambda}_1 = \sqrt{\gamma\Lambda_1}$ (note that $0 < \lambda_1 \leqslant b$, $0 < \tilde{\lambda}_1 \leqslant \tilde{b}$), and proceed as follows. We define Herglotz functions $\phi_0, \tilde{\phi}_0$ by

$$\phi_0 = h_{b,\lambda_1} \circ \Phi_0 \circ h_{b,\lambda_1}^{-1}, \qquad \tilde{\phi}_0 = h_{\tilde{b},\tilde{\lambda}_1} \circ \tilde{\Phi}_0 \circ h_{\tilde{b},\tilde{\lambda}_1}^{-1}. \tag{5.63}$$

The functions $\phi_0, \tilde{\phi}_0$ are analytic in the domains $\Omega(-\lambda_1^{-1}, a_2(\lambda_1)^{-1}), \Omega(-\tilde{\lambda}_1^{-1}, \tilde{a}_2(\tilde{\lambda}_1)^{-1})$ respectively, where $a_2, \tilde{a}_2$ were defined in (5.24), (5.25) respectively, and map these domains into themselves. ϕ_0 and $\tilde{\phi}_0$ possess the same properties as in the previous subsection, except for

$$\phi_0'(1) = \Lambda_0^r, \qquad \tilde{\phi}_0'(1) = \tilde{\Lambda}_0^r.$$

The linearizers $\psi_1, \tilde{\psi}_1$ are the unique anti-Herglotz functions analytic in $\Omega\big(-\tilde{\lambda}_1^{-1}, \tilde{a}_2(\tilde{\lambda}_1)^{-1}\big)$, $\Omega\big(-\lambda_1^{-1}, a_2(\lambda_1)^{-1}\big)$ respectively, such that

$$\psi_1(z) = \frac{1}{\tilde{\Lambda}_0^r}\psi_1(\tilde{\phi}_0(z)), \qquad z \in \Omega(-\tilde{\lambda}_1^{-1}, \tilde{a}_2(\tilde{\lambda}_1)^{-1}), \quad \psi_1(0) = 1, \quad \psi_1(1) = 0, \tag{5.64}$$

$$\tilde{\psi}_1(z) = \frac{1}{\Lambda_0^r}\tilde{\psi}_1(\phi_0(z)), \qquad z \in \Omega(-\lambda_1^{-1}, a_2(\lambda_1)^{-1}), \quad \tilde{\psi}_1(0) = 1, \quad \tilde{\psi}_1(1) = 0. \tag{5.65}$$

They satisfy

$$\psi_1\big(-\tilde{\lambda}_1^{-1}\big) = \frac{1}{\tilde{\Lambda}_0^r}\psi_1\big(\tilde{\phi}_0\big(-\tilde{\lambda}_1^{-1}\big)\big) \leqslant \frac{1}{\tilde{\Lambda}_0^r},$$
$$\tilde{\psi}_1\big(-\lambda_1^{-1}\big) = \frac{1}{\Lambda_0^r}\tilde{\psi}_1\big(\phi_0\big(-\lambda_1^{-1}\big)\big) \leqslant \frac{1}{\Lambda_0^r}.$$

In the preceding subsection the bounds $\psi(-\tilde{\lambda}^{-1}) \leqslant \Lambda^{-r}$, $\tilde{\psi}(-\lambda^{-1}) \leqslant \Lambda^{-r}$ were important. To restore analogous bounds in the present situation, we define new functions $\psi, \tilde{\psi}$ by

$$\psi = \theta \circ \psi_1, \qquad \tilde{\psi} = \tilde{\theta} \circ \tilde{\psi}_1, \tag{5.66}$$

where $\theta, \tilde{\theta}$ are Herglotz functions given by

$$\theta(z) = \begin{cases} z, & \Lambda_1 \leqslant \tilde{\Lambda}_0, \\ \theta_1(z) = \dfrac{z(1-\tilde{\Lambda}_0^r)}{z(\Lambda_1^r - \tilde{\Lambda}_0^r) + 1 - \Lambda_1^r}, & \Lambda_1 > \tilde{\Lambda}_0, \end{cases} \tag{5.67}$$

$$\tilde{\theta}(z) = \begin{cases} z, & \Lambda_1 \leqslant \Lambda_0, \\ \tilde{\theta}_1(z) = \dfrac{z(1-\Lambda_0^r)}{z(\Lambda_1^r - \Lambda_0^r) + 1 - \Lambda_1^r}, & \Lambda_1 > \Lambda_0. \end{cases} \tag{5.68}$$

These functions fix 0 and 1, $\theta_1\big(\tilde{\Lambda}_0^{-r}\big) = \Lambda_1^{-r}$, $\tilde{\theta}_1\big(\Lambda_0^{-r}\big) = \Lambda_1^{-r}$, and they have poles at negative values. Let κ denote the pole of θ_1 (and θ). Then $\psi \in AH\big(-\tilde{\lambda}_1^{-1}, \tilde{a}_3^{-1}\big)$, where $\tilde{a}_3^{-1} = \psi_1^{-1}(\kappa)$ if $\kappa \in \psi_1((1, \tilde{a}_2(\tilde{\lambda}_1)^{-1})$ and $\tilde{a}_3^{-1} = \tilde{a}_2(\tilde{\lambda}_1)^{-1}$ otherwise. For $z > 1$, $\psi_1(z) < 0$ and from inequalities (4.4) we have

$$\psi_1(z) \geqslant \psi_2(z) = \frac{1-z}{1-\tilde{a}_2(\tilde{\lambda}_1)z}.$$

If $y = \psi_1^{-1}(\kappa) < \tilde{a}_2(\tilde{\lambda}_1)^{-1}$ then, since ψ_2 is decreasing, we have

$$\kappa = \psi_1(y) \geqslant \psi_2(y), \qquad \psi_2^{-1}(\kappa) \leqslant y.$$

Thus ψ is analytic in $\Omega\big(-\tilde{\lambda}_1^{-1}, \tilde{\ell}^{-1}\big)$, where $\tilde{\ell}^{-1} = \psi_2^{-1}(\kappa) = \psi_2(\kappa)$. Using $\kappa = -\big(1-\Lambda_1^r\big)/\big(\Lambda_1^r - \tilde{\Lambda}_0^r\big)$ this gives

$$\begin{aligned} \tilde{\ell} &= \frac{\Lambda_1^r - \tilde{\Lambda}_0^r + \big(1-\Lambda_1^r\big)\tilde{a}_2(\tilde{\lambda}_1)}{1-\tilde{\Lambda}_0^r}, \\ \tilde{a}(\Lambda_1) &\leqslant \tilde{a}_2(\tilde{\lambda}_1) \leqslant \tilde{\ell} < \tilde{a}_4(\tilde{\lambda}_1) = \Lambda_1^r + \big(1-\Lambda_1^r\big)\tilde{a}_2(\tilde{\lambda}_1). \end{aligned} \tag{5.69}$$

Similarly, $\tilde{\psi} \in AH(-\lambda_1^{-1}, \ell^{-1})$, where

$$\begin{aligned} \ell &= \frac{\Lambda_1^r - \Lambda_0^r + \big(1-\Lambda_1^r\big)a_2(\lambda_1)}{1-\Lambda_0^r}, \\ a(\Lambda_1) &\leqslant a_2(\lambda_1) \leqslant \ell < a_4(\lambda_1) = \Lambda_1^r + \big(1-\Lambda_1^r\big)a_2(\lambda_1). \end{aligned} \tag{5.70}$$

The functions $\psi, \tilde{\psi}$ have been defined so as to satisfy $\psi\big(-\tilde{\lambda}_1^{-1}\big) \leqslant \Lambda_1^{-r}$, $\tilde{\psi}(-\lambda_1^{-1}) \leqslant \Lambda_1^{-r}$. We now proceed to define $v, \tilde{v}, z_1, \tilde{z}_1, \phi, \tilde{\phi}$, etc, exactly as in the preceding subsection and obtain the same estimates with the only exception the lower bounds for $\phi'(1)$ and $\tilde{\phi}'(1)$.

Suppose $\lambda_1 < \tilde{\lambda}_1$. The case $\lambda_1 > \tilde{\lambda}_1$ is similar. We must set $\tilde{c} = \tilde{a}_4(\tilde{\lambda}_1)$ in (5.50), and since $\lambda = \lambda_1, \tilde{\lambda} = \tilde{\lambda}_1$ we find

$$\phi'(1) \geqslant \lambda_1^p\, l(\lambda_1, \tilde{\lambda}_1) = \lambda_1^p \left(\frac{1-\tilde{a}_4(\tilde{\lambda}_1)}{r(1+\lambda_1)(1+\tilde{a}_4(\tilde{\lambda}_1)\lambda_1)}\right)\left(\frac{1-\tilde{a}_4(\tilde{\lambda}_1)}{r(\lambda_1+\tilde{a}_4(\tilde{\lambda}_1))(1+\lambda_1)}\right)^{p-1}. \tag{5.71}$$

Similarly we obtain the bound

$$\tilde{\phi}'(1) \geqslant \tilde{\lambda}_1\lambda_1^{p-1}\tilde{l}(\lambda_1, \tilde{\lambda}_1) = \tilde{\lambda}_1\lambda_1^{p-1}\left(\frac{1-a_4(\lambda_1)}{r(1+\tilde{\lambda}_1)(1+a_4(\lambda_1)\tilde{\lambda}_1)}\right)\left(\frac{1-a_4(\lambda_1)}{r(1+\lambda_1)(\lambda_1+a_4(\lambda_1))}\right)^{p-1}. \tag{5.72}$$

Writing $\lambda_1 = \sqrt{\frac{\Lambda_1}{\gamma}}$, $\tilde\lambda_1 = \sqrt{\gamma\Lambda_1}$ we have from (5.71) and (5.72)

$$\phi'(1)\tilde\phi'(1) \geqslant \mathcal{L}(\Lambda_1,\gamma) = \Lambda_1^p\gamma^{1-p}\, l(\Lambda_1,\gamma)\,\tilde l(\Lambda_1,\gamma). \tag{5.73}$$

Recall that $\phi, \tilde\phi$ are analytic in $\Omega(-\lambda^{-1}, a(\Lambda)^{-1})$, $\Omega(-\tilde\lambda^{-1}, \tilde a(\Lambda)^{-1})$ respectively and map these domains into themselves, with $a, \tilde a$ given by (5.21), (5.22), respectively. We note that the bounds (5.71), (5.72) also hold in the cases when $\Lambda = \min\{\Lambda_0, \tilde\Lambda_0\} \geqslant \Lambda_1$, and when $\theta_1(z) = z, \tilde\theta_1(z) = z$, since $\tilde a_2(\tilde\lambda) \leqslant \tilde a_2(\tilde\lambda_1) < \tilde a_4(\tilde\lambda_1)$, and $a_2(\lambda) \leqslant a_2(\lambda_1) < a_4(\lambda_1)$.

Finally we define

$$N(\Phi_0, \tilde\Phi_0) = \left(h^{-1}_{b,\lambda_1}\circ\phi\circ h_{b,\lambda_1},\; h^{-1}_{\tilde b,\tilde\lambda_1}\circ\tilde\phi\circ h_{\tilde b,\tilde\lambda_1}\right). \tag{5.74}$$

The operator $N(r,p,\gamma,\Lambda_1)$ maps the domain $\tilde{\mathcal{Q}}_2(r,\gamma,\Lambda_+)$ into $\tilde{\mathcal{Q}}_2(r,\gamma,\Lambda_+)\cap\{(\Phi,\tilde\Phi) : \Phi'(1)\tilde\Phi'(1) \geqslant \mathcal{L}(\Lambda_1,\gamma)\}$ which is convex and compact and therefore it has fixed points there.

5.3.2. Fixed points of N. We now have to show that if Λ_1 has been chosen sufficiently small, any fixed point of N is actually a fixed point of B. We assume, from now on, that $\Lambda_1 \leqslant \Lambda_\star$. Let $(\Phi_0, \tilde\Phi_0)$ be a fixed point of N. If $\min\{\Lambda_0, \tilde\Lambda_0\} \geqslant \Lambda_1$, there is nothing to prove. Otherwise, we have $\Lambda = \Lambda_1$, $\phi_0 = \phi$ and $\tilde\phi_0 = \tilde\phi$, so that $\tilde\phi_0, \psi_1$ are now analytic in $\Omega(-\tilde\lambda_1^{-1}, \tilde a(\Lambda_1)^{-1})$, and $\phi_0, \tilde\psi_1$ are now analytic in $\Omega(-\lambda_1^{-1}, a(\Lambda_1)^{-1})$. Thus, the function ψ is now analytic in $\Omega(-\tilde\lambda_1^{-1}, \tilde a_5(\Lambda_1)^{-1})$, where

$$\tilde a(\Lambda_1) < \tilde a_5(\Lambda_1) = \frac{\Lambda_1^r - \tilde\Lambda_0^r + \left(1-\Lambda_1^r\right)\tilde a(\Lambda_1)}{1-\tilde\Lambda_0^r} < \Lambda_1^r + \left(1-\Lambda_1^r\right)\tilde a(\Lambda_1). \tag{5.75}$$

Recalling that $\Lambda_1 \leqslant \Lambda_\star$, we have

$$\tilde a_5(\Lambda_1) \leqslant \Lambda_1^r + \left(1-\Lambda_1^r\right)\frac{(1+\sqrt{\gamma})\sqrt{\Lambda_1}}{1-\sqrt{\Lambda_1}} \leqslant \frac{(2+\sqrt{\gamma})\sqrt{\Lambda_1}}{1-\sqrt{\Lambda_1}} = \frac{(2+\sqrt{\gamma})\sqrt{\gamma}\lambda_1}{1-\sqrt{\gamma}\lambda_1}. \tag{5.76}$$

Suppose that $\lambda_1 < \tilde\lambda_1$. The case $\lambda_1 > \tilde\lambda_1$ is similar. Using (5.76) in the lower bound obtained by setting $\lambda = \lambda_1$ and $\tilde c = \tilde a_5(\Lambda_1)$ in (5.50) we find

$$\phi'(1) \geqslant \lambda_1\left(\frac{1-3\sqrt{\gamma}\lambda_1-\gamma\lambda_1}{r(1+\lambda_1)\left(1-\sqrt{\gamma}\lambda_1+(2+\sqrt{\gamma})\sqrt{\gamma}\lambda_1^2\right)}\right)$$
$$\times\left(\frac{1-3\sqrt{\gamma}\lambda_1-\gamma\lambda_1}{r(1+\lambda_1)(1+2\sqrt{\gamma}+\gamma-\sqrt{\gamma}\lambda_1)}\right)^{p-1}.$$

Writing λ_1 in terms of Λ_1, and using $\Lambda_1 \leqslant \Lambda_\star$, we obtain

$$\phi'(1) \geqslant \lambda_1\left(\frac{\sqrt{\gamma}(1-(3+\sqrt{\gamma})\sqrt{\Lambda_\star})}{r(\sqrt{\gamma}+\sqrt{\Lambda_\star})}\right)^p\frac{\sqrt{\gamma}}{(\sqrt{\gamma}+(2+\sqrt{\gamma})\Lambda_\star)(1+\sqrt{\gamma})^{2(p-1)}}$$
$$= \lambda_1 K_4(r,p,\gamma,\Lambda_\star). \tag{5.77}$$

Also, $\tilde\psi$ is analytic in $\Omega(-\lambda_1^{-1}, a_5(\Lambda_1)^{-1})$, where

$$a(\Lambda_1) < a_5(\Lambda_1) = \frac{\Lambda_1^r - \Lambda_0^r + \left(1-\Lambda_1^r\right)a(\Lambda_1)}{1-\Lambda_0^r} < \Lambda_1^r + \left(1-\Lambda_1^r\right)a(\Lambda_1) \leqslant \frac{(1+2\sqrt{\gamma})\sqrt{\Lambda_1}}{\sqrt{\gamma}(1-\sqrt{\Lambda_1})}, \tag{5.78}$$

and one can verify the bound, in the case $\lambda_1 < \tilde\lambda_1$,

$$\tilde\phi'(1) \geqslant \tilde\lambda_1\tilde K_4(r,p,\gamma,\Lambda_\star). \tag{5.79}$$

Therefore, from (5.77) and (5.79) we have

$$\phi'(1)\tilde\phi'(1) \geqslant \lambda_1\tilde\lambda_1 K_4(r,p,\gamma,\Lambda_\star)\tilde K_4(r,p,\gamma,\Lambda_\star) = \Lambda_1 K_4\tilde K_4. \tag{5.80}$$

The fact that $(\Phi_0, \tilde{\Phi}_0)$ is a fixed point implies that $\Phi_0'(1)^{\frac{1}{r}} = \Lambda_0 = \phi'(1)^{\frac{1}{r}}$, and $\tilde{\Phi}_0'(1)^{\frac{1}{r}} = \tilde{\Lambda}_0 = \tilde{\phi}'(1)^{\frac{1}{r}}$. Since $\min\{\Lambda_0, \tilde{\Lambda}_0\} < \Lambda_1$, we have $\Lambda_1 > \Lambda_0$ or $\Lambda_1 > \tilde{\Lambda}_0$ (or both), so that $\Lambda_1 > \Lambda_0\tilde{\Lambda}_0$ as $\Lambda_0 < 1$, $\tilde{\Lambda}_0 < 1$. Hence $\Lambda_1 > \phi'(1)^{\frac{1}{r}}\tilde{\phi}'(1)^{\frac{1}{r}}$. Using this in (5.80) we find

$$\phi'(1)\tilde{\phi}'(1) \geqslant (K_4\tilde{K}_4)^{\frac{r}{r-1}}. \tag{5.81}$$

If we assume that Λ_1 has been chosen so that $\Lambda_1 < (K_4\tilde{K}_4)^{\frac{1}{r-1}}$, inequality (5.81) contradicts our hypothesis that $\min\left\{\Phi_0'(1)^{\frac{1}{r}}, \tilde{\Phi}_0'(1)^{\frac{1}{r}}\right\} < \Lambda_1$, i.e. $\Phi_0'(1)\tilde{\Phi}_0'(1) < \Lambda_1^r$. Therefore $(\Phi_0, \tilde{\Phi}_0)$ is a fixed point of B.

6. Conclusion

In this paper we have investigated a class of functional equations closely related to the period-two renormalization equations for non-commuting critical circle-map pairs. Using the Herglotz function technique of H Epstein, we have obtained the existence of analytic solutions of these equations. However, in order for the theory to correspond to that of circle-map pairs, an additional condition is required which we have yet to establish with this theory. The obstacle is one of the techniques rather than fundamental, and we are confident that further work will establish the precise theory required for the application to non-commuting circle-map pairs.

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